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Structure of the conservation laws in integrable spin chains with short range interactions

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Abstract

We present a detailed analysis of the structure of the conservation laws in quantum integrable chains of the XYZ-type and in the Hubbard model. The essential tool for the former class of models is the boost operator, which provides a recursive way of calculation of the integrals of motion. With its help, we establish the general form of the XYZ conserved charges in terms of simple polynomials in spin variables and derive recursion relations for the relative coefficients of these polynomials. Although these relations are difficult to solve in general, a subset of the coefficients can be determined. Moreover, for two submodels of the XYZ chain - namely the XXX and XY cases, all the charges can be calculated in closed form. Using this approach, we rederive the known expressions for the XY charges in a novel way. For the XXX case, a simple description of conserved charges is found in terms of a Catalan tree. This construction is generalized for the $su(M)$ invariant integrable chain. We also investigate the circumstances permitting the existence of a recursive (ladder) operator in general quantum integrable systems. We indicate that a quantum ladder operator can be traced back to the presence of a hamiltonian mastersymmetry of degree one in the classical continuous version of the model. In this way, quantum chains endowed with a recursive structure can be identified from the properties of their classical relatives. We also show that in the quantum continuous limits of the XYZ model, the ladder property of the boost operator disappears. For the Hubbard model we demonstrate the non-existence of a ladder operator. Nevertheless, the general structure of the conserved charges is indicated, and the expression for the terms linear in the model's free parameter for all charges is derived in closed form.

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1. Introduction

This work is concerned with a detailed analysis of the structure of the conservation laws for quantum integrable chains with short range interactions. Attention is focused on the XYZ model:

$$H = \sum_{j\in\Lambda} [\lambda_x \sigma_j^x \sigma_{j+1}^x + \lambda_y \sigma_j^y \sigma_{j+1}^y + \lambda_z \sigma_j^z \sigma_{j+1}^z], \tag{1.1}$$

(where Λ is the spin lattice, $\sigma_i^{x,y,z}$ are the Pauli sigma matrices, acting non-trivially only on the i-th site, and $\lambda_x, \lambda_y, \lambda_z$ are constants), the Hubbard model:

$$H = \sum_{j\in\Lambda} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \tau_j^x \tau_{j+1}^x + \tau_j^y \tau_{j+1}^y + U \sigma_j^z \tau_j^z], \tag{1.2}$$

(with σ and τ standing for two independent sets of sigma matrices, and U a constant), and their close relatives. A brief survey of known results precedes the presentation of our findings.

1.1. The problem of calculating conservation laws and ladder operators in integrable systems

Beyond the existence proofs, very few explicit results are known on the structure of the conserved charges of integrable quantum chains, even for rather basic systems such as the XYZ or the Hubbard model. Most of the results concerning these two particular classes of models were obtained by means of the Bethe ansatz method, whose immediate concern is the construction of the hamiltonian eigenstates (see [1] for the XYZ-type models, and [2] for the Hubbard model). For the XYZ model, a second input came from the discovery of its equivalence with an exactly solvable two-dimensional classical statistical model, the eight-vertex model [3]. The row-to-row transfer matrix of the latter model, denoted by $T(\lambda)$, where λ is a spectral parameter arising in the parameterization of the Boltzmann weights, turns out to commute with the XYZ hamiltonian [4]:

$$[T(\lambda), H] = 0. \tag{1.3}$$

The expansion of the logarithm of the transfer matrix in terms of the spectral parameter yields a family of conserved charges. This family is infinite in the infinite chain limit. Furthermore, the row-to-row transfer matrices evaluated for different values of the spectral parameter were shown to commute [5]:

$$[T(\lambda), T(\mu)] = 0. \tag{1.4}$$

This directly implies the mutual commutativity of all these conserved charges and thereby proves the complete integrability of the XYZ chain.

The last statement embodies implicitly the standard working definition of integrability for a quantum system with N degrees of freedom, namely the existence of N independent, mutually commuting integrals of motion. This aspect (the existence of a family of commuting charges) of a system that can be solved by the Bethe ansatz in its original (coordinate) version is not immediate, although non-questionable. In the context of the Hubbard model, the absence of a direct link between the coordinate Bethe ansatz and integrability, motivated Shastry [6] to parallel the integrability proof of the XYZ model by constructing the two-dimensional classical statistical version of the Hubbard model, whose transfer matrix is related to the Hubbard hamiltonian. The demonstration of (1.4) for this transfer matrix establishes then "directly" the integrability of the Hubbard model. It should be added that in the quantum inverse scattering reformulation of the Bethe ansatz (the algebraic Bethe ansatz) developed in the last 15 years [7, 8], the transfer matrix is a central object, which makes algebraic Bethe ansatz solvability and quantum integrability in the above sense, essentially equivalent. However, the Hubbard model has not yet be reformulated from that point of view.

Recent works on non-abelian symmetries in quantum chains call for a clarifying comment concerning the relation between the transfer matrix and the conserved charges. The expansion of the logarithm of the transfer matrix in terms of the spectral parameter at $\lambda = 0$ (or some other suitable finite value) leads to *local* conserved integrals of motion, which all mutually commute [9]. The first member of this infinite sequence is proportional to the defining hamiltonian of the quantum chain. Locality means that interaction involving a certain set of sites disappears when the distances between them become sufficiently large. On the other hand, the expansion of the monodromy matrix at $\lambda = \infty$ produces conserved integrals which are *non-local* and *non-commuting* (see e.g. the review [10] which focuses on the isotropic case, i.e. the XXX model). More precisely, these charges all commute with the hamiltonian and the higher local charges, but not among themselves. They in fact generate a highly non-trivial non-abelian algebra. For the XXX model for instance, this algebra provides a realization of a Yangian [11]. Notice moreover that the commutativity of these non-local charges with the local ones is true only in the limit of an infinite chain, in contrast with the local ones, which are conserved also for finite chains with periodic boundary conditions. In this work, we are mainly interested in the local charges and we will have little to say about the non-local ones.

The most direct way of proving the integrability of a quantum chain, independently of (any) Bethe-ansatz formalism or a relation with a two-dimensional statistical system, is certainly to display a quantum Lax equation that reproduces the hamiltonian equation of motion of the chain. This construction can be done rather systematically. For the XYZ chain and the Hubbard model, the results can be found respectively in [12, 13] and [14]. However, this framework does not lead to a new way of calculating the conserved integrals. Indeed, the Lax operator depends upon a spectral parameter and the conserved laws are recovered from a series expansion, in terms of this spectral parameter, of the product of all the Lax operators, one for each site. This product is nothing but the transfer matrix.

Therefore, at our present state of knowledge, a systematic computation of the conserved integrals in a generic quantum integrable chain must necessarily proceed through the series expansion of the generating function, i.e. the transfer matrix. But for actual computations, this method is completely impractical, apart from the evaluation of the hamiltonian itself and maybe a first few non-trivial conserved laws. The reason for this is that the transfer matrix is a formidable object, with its size growing exponentially with the number of spins. Even relatively small chains ($|\Lambda| \sim 10$) provide a difficult challenge for computer algebra programs, requiring vast amounts of running time and memory.

A different approach is thus clearly desirable. Fortunately, in the XYZ model there exist an alternative, recursive way of calculating the conserved charges. It is based on the existence of a ladder operator, i.e. an operator B with the property:

$$[B, Q_n] \sim Q_{n+1}, \tag{1.5}$$

where Q_n denotes a charge with at most n adjacent spins interacting, with Q_2 being proportional to the hamiltonian.[1] For the XYZ model, such an operator is provided by the first moment of the hamiltonian density (see e.g. [15, 16]); i.e., if one writes the hamiltonian in the form

$$H \sim \sum_{j \in \Lambda} h_{j,j+1}, \tag{1.6}$$

it reads

$$B = \sum_{j \in \Lambda} j \; h_{j,j+1}. \tag{1.7}$$

[1] Note that we allow for the possibility of a linear combination of lower order charges $Q_{m \leq n}$ on the (rhs) of 1.5.

The operator B generates Lorentz transformations, by acting on the transfer matrix by differentiation with respect to the spectral parameter [17, 18]:

$$[B, T(\lambda)] = \frac{\partial T(\lambda)}{\partial \lambda}. \tag{1.8}$$

Due to (1.8), B is usually called a boost operator. For models with short-range interaction, which are related to a statistical system via the transfer matrix formalism, both (1.8) and the ladder property (1.5) are equivalent. But since we emphasize the latter property, we will most often use the more appropriate ladder qualitative.

The formula (1.5) is a useful tool. Not only does it provide a convenient recursive way to calculate explicit expressions for the conserved charges, avoiding the computational difficulties with the expansion of the transfer matrix, but, as we will see below, it is also helpful in proving general properties of the family $\{Q_n\}$. However, even with the use of (1.5), the complexity of calculations of higher order charges is still considerable. Moreover, the problem is not only in the computational complexity, but also that the results, at first sight, seem hopelessly complicated and no pattern can be easily discerned. In one of the earlier works concerned with calculations of the explicit form[2] of the conservation laws [21], the charge Q_6 is described in the following way: [it] "is really a monster representing interactions of up to 6 neighbors and being a sum of roughly 100 terms (each being an infinite sum). A good guess is that the explicit form of this hamiltonian will never appear explicitly in the literature."[3]

It is interesting to investigate the general circumstances permitting the existence of a ladder operator in integrable systems. For classical integrable systems, the most common recursive structure is the one related to their bi-hamiltonian character (see e.g. [22]). Recall that the existence of *two* distinct hamiltonian structures is an almost universal signature of integrability is for classical hamiltonian models. Take for instance a continuous system described by a set of fields collectively denoted by φ. The bi-hamiltonian character of the system translates into the equalities

$$\partial_t \varphi = \{\varphi, H\}_{(1)} = P_1 \frac{\delta H}{\delta \varphi} = \{\varphi, H'\}_{(2)} = P_2 \frac{\delta H'}{\delta \varphi}, \tag{1.9}$$

[2] The three-spin charge Q_3 for the XYZ model was calculated in [9]. The explicit form of the four-spin charge appears in [19]. For the isotropic (XXX) model it can be also obtained, as a special case, from the four-point charge of the t-J model, written down in [20].

[3] But see section (5.3)!

where H, H' are two distinct hamiltonians, and P_i $(i = 1, 2)$ denotes the differential operator, defining the equal-time Poisson bracket $\{\ ,\ \}_{(i)}$ via

$$\{\varphi(x), \varphi(y)\}_{(i)} = P_i(x)\delta(x - y). \tag{1.10}$$

If H' has higher order than H in terms of an appropriate grading, it follows that the charge H_{n+1} can be calculated from the lower order one H_n by

$$P_1^{-1} P_2 \frac{\delta H_n}{\delta \varphi} = \frac{\delta H_{n+1}}{\delta \varphi}. \tag{1.11}$$

The bi-hamiltonian structure furnishes then a recursion operator, $P_1^{-1}P_2$, that acts, modulo functional derivatives, as a sort of ladder operator on the conserved charges.

Constructing the quantum version of a classical integrable hamiltonian system amounts to a quantization of either of the two Poisson structures. The point is that the quantization of the system via P_1 gives a theory which, in general is quite distinct from the one obtained from the quantization of P_2. Hence, at the quantum level, the bi-hamiltonian character is irremediably lost.[4] As a consequence, a priori there are no reasons to expect a recursive scheme for the calculation of the conserved charges in quantum integrable systems.

As we just argued, the ladder property (1.5) does not reflect a sort of accidental bi-hamiltonian structure: the action of B is clearly not a direct quantum generalization of (1.11). The classical counterpart of (1.5) would be

$$\{B, H_n\}_{(i)} = H_{n+1}, \tag{1.12}$$

for some Poisson structure $\{\ ,\ \}_{(i)}$ corresponding to the commutator appearing in (1.5). This equation has a simple, but somewhat unfamiliar interpretation in soliton theory: it reflects the presence of a *hamiltonian mastersymmetry of degree 1*, which we denote by τ_1. For systems whose evolution equation takes the form

$$\varphi_t = K(\varphi), \tag{1.13}$$

a mastersymmetry τ_1 is defined by

$$[[\tau_1, K(\varphi)], K(\varphi)] = 0, \tag{1.14}$$

[4] This has apparently first been observed in [23] in the context of the quantum Korteweg-de Vries equation.

(where the commutators should be understood as Lie derivatives). A mastersymmetry is hamiltonian if it can be written as

$$\tau_1 = P_i \frac{\delta H_n}{\delta \varphi} \tag{1.15}$$

for some H_n. If P_i is a fundamental hamiltonian operator, meaning that it cannot be factorized into a product of powers of other hamiltonian operators, the existence of a hamiltonian mastersymmetry of degree 1 is the exception rather than the rule.[5] But in the Landau-Lifshitz model, which corresponds to a continuous, classical version of the XYZ model, such a mastersymmetry does exist [24]. This, in a sense, is the reason for the existence of a ladder operator for the XYZ quantum chain.

In the XYZ context, the boost property (1.8) is actually rooted in the Yang-Baxter equation itself, or more precisely, in the fact that the transfer matrix is a product of the R matrices, satisfying the Yang-Baxter equation [17]. In other words, the Lax operator and the R matrix are identical. In the terminology of [8], such models are said to be fundamental. Somewhat paradoxically, that relation between the boost operator and the Yang-Baxter equation might suggest that the existence of a quantum ladder operator is a general feature of integrable quantum chains! However, since not all integrable spin models are fundamental, the existence of a ladder operator B satisfying (1.5) cannot be the hallmark of quantum integrable chains. In particular, no such operator exists for the Hubbard model (as shown below); this model is indeed not fundamental in the above sense. There are other examples of integrable chains without ladder operator. For instance, for the trigonometric (or hyperbolic) long-range interaction version of the XXX model presented by Haldane and Shastry [25], no such operator is known.[6]

It is of great interest to study the continuous limit of the ladder operator B. Indeed, in appropriate sectors, ferromagnetic or antiferromagnetic, the XYZ model reduces either to the quantum nonlinear Schrödinger equation or the Thirring model (equivalent to the sine-Gordon equation by bosonization, itself related to the quantum Korteweg-de Vries equation by a Miura-Feigen-Fuchs transformation). Although the quantum inverse

[5] This statement will be made more precise in appendix A, where the concept of mastersymmetry is also briefly reviewed. At this point we simply mention that (1.15) is not verified for most bi-hamiltonian systems if P_i a fundamental hamiltonian operator.

[6] The original argument establishing the existence of B for the XYZ model, which is reviewed in section (2), breaks down for a chain with long-range interactions.

scattering method furnishes a systematic method for calculating the conserved charges of these theories, it is again rather impractical. Hence, the existence of B in the XYZ model gives a hope that a ladder operator could exist also in these continuous models. However, as we will show, this hope does not materialize.

1.2. An overview of the present results

Our aim in this work is to unravel the general pattern of the conserved integrals. That amounts to first determining which terms can appear in the charge of a given order, and eventually fixing the values of their coefficients. Our strategy is rather straightforward: from the explicit expression of the first few conserved integrals, we identify the general pattern, which is subsequently established by means of a recursive argument. In some favorable circumstances, all the coefficients can be calculated explicitly. This occurs for two special cases of the XYZ model: the XXX model ($\lambda_x = \lambda_y = \lambda_z$ in (1.1)) and the degenerate XY model ($\lambda_z = 0$). The conserved quantities in the latter case are well-known (the model being equivalent to a free fermion theory) and have been presented in [26, 27, 28, 29]. For the infinite XXX model, a general expression for the conserved integrals was first reported (without proof) in [30]. Originally unaware of this work, we have rediscovered it in a modified form, presented in [31], with a sketch of the proof. In this work we give the details of the the proof. We also extend this construction to the isotropic, $su(M)$ invariant generalizations of the XXX chain. In the general anisotropic case of the $s = 1/2$ XYZ model, the functional form of the integrals of motion is the same as in the isotropic case. However, the coefficients become rather complicated polynomials in the coupling constants. We have been able to determine exactly only some of these coefficients. No significant simplification of the XYZ conservation laws has been found in the special XXZ case ($\lambda_x = \lambda_y$).

In the absence of a ladder operator, as is the case for the Hubbard model, the general form of the conserved law must first be guessed, with each term multiplied by an undetermined coefficient. These are then fixed by enforcing the commutativity with the hamiltonian. Fortunately, for the Hubbard model, a simple pattern soon emerges, in which the building blocks of the conserved densities are recognized as the XX conserved densities (which is not surprising since the Hubbard model is essentially two copies of the XX model interacting along their z-component.) This observation significantly reduces the number of a priori undetermined free parameters. With the normalization we have chosen for the parameter U (cf. eq. (1.2)), all coefficients turn out to be simply $\pm U^\ell$, with ℓ a non-negative

integer. Furthermore, the different terms in H_n can be grouped into classes, where all the terms of a given class have the same coefficient and can be obtained from each other by a relative lattice translation of its components; these classes are in correspondence with the partitions of the integers $n - 2r \geq 0$, where r is again a non-negative integer. The coefficients of the terms independent of U and those linear in U are found exactly.[7]

Appendices are devoted to the study of some continuous theories related to the XYZ model. Appendix A is concerned with a general discussion of mastersymmetries in classical soliton theory, followed by a presentation of the results pertaining to the classical Landau-Lifshitz equation. Although this appendix is mainly composed with review material, it contains several novel observations. In particular, it determines the conditions under which classical mastersymmetries and a quantum boost operator can be related. This provides a way of detecting classical integrable models whose quantum version will contain a ladder operator. Next, by a direct analysis presented in the appendix B, we establish the evanescent nature of the ladder operator in the continuous limit of the XYZ model, by disproving the possible existence of ladder operators both for the quantum nonlinear Schrödinger and the quantum Korteweg-de Vries equations.

Many of the calculations of the higher order charges in this work, which are often quite demanding computationally, were performed using a set of specialized "Mathematica" routines that we developed.

1.3. Motivation

What is the motivation for constructing explicitly the conserved charges of the quantum integrable chains? First of all, it forces us to look at these systems from a point of view different from the Bethe ansatz. New perspectives can obviously lead to a deeper understanding of these models and more generally of the nature of quantum integrable systems, for which, after all, relatively little is known.

Another motivation is related to model building. It is always of interest to be able to mimic the gross features of a physical problem in terms of an integrable system. A simple way of producing new integrable physical models in the context of an integrable

[7] This implies the knowledge of the explicit form of all conservation laws of a hypothetical physical model that would be described by the Hubbard hamiltonian but with a parameter U satisfying $U^2 = 0$, a constraint that does not spoil the integrability of the model. Such a constraint can be realized e.g. by having U of the form $U = \alpha\beta$, with α and β constant Grassmannian parameters.

hierarchy, is to take as the basic hamiltonian, some linear combinations of the conserved charges, choosing the precise combination that produces the desirable dynamical aspects of the problem under consideration. The simplest illustration of this idea is the description of the XXZ model in a magnetic field. It is obtained by adding to the original hamiltonian, a term proportional to the lowest order conserved charge, namely the z-component of the total spin, which is multiplied by a constant coefficient having the physical interpretation of an external magnetic field. Frahm has proposed recently a more sophisticated realization of this program, by considering a XXZ-type model with basic hamiltonian $Q_2 + cQ_3$ [32].

The knowledge of explicit formulae for the conserved charges could also be useful in finding exact solutions for small finite chains (see e.g. [33]).

Our original motivation for this project was to see to what extent information on the conserved charges of the continuous quantum integrable systems could be extracted from those of their lattice regularization. In particular, if the explicit formulae for all the XXX conserved charges are known, does it mean that all the conserved charges of the corresponding continuous theory are also known? However, the continuous limit of the higher order charges is not straightforward. The problem is essentially that products of operators in the continuous theory have to be regularized in some way (e.g. via normal order) and no clear print of this regularization appears to be encoded in the lattice version. A full discussion of these questions is deferred to a subsequent work.

2. The Yang-Baxter equation, commuting transfer matrices and the boost operator

We first briefly review the relation between the conserved charges of a quantum spin chain with the commuting family transfer matrices and the Yang-Baxter equation. Since we do not consider in this work non-trivial boundary effects, the following discussion is confined to infinite chains or finite chains with periodic boundary conditions.

Consider a quantum chain with some spin-like variables S_i^a, $a = 1, ..., d$, defined at site i and valued in some Lie algebra, which thereby fixes the defining commutation relation of the model. The dynamics is governed by a hamiltonian H containing only nearest-neighbor interaction terms. The canonical equation of motion

$$\frac{d\mathbf{S}_i}{dt} = [\mathbf{S}_i, H] \tag{2.1}$$

is supposed to be equivalent to the compatibility condition for the linear system

$$\begin{aligned} \frac{d\mathbf{S}_i(t)}{dt} =& U_i(\lambda)\mathbf{S}_i, \\ \mathbf{S}_{i+1}(t) =& L_i(\lambda)\mathbf{S}_i, \end{aligned} \tag{2.2}$$

which reads

$$\frac{dL_i}{dt} = U_{i+1}L_i - L_iU_i. \tag{2.3}$$

U and L are $d \times d$ matrices depending upon a complex spectral parameter λ. Finally, the Lax operator L is assumed to satisfy the following relation:

$$\check{R}(\lambda - \mu)(L_i(\lambda) \otimes L_i(\mu)) = (L_i(\mu) \otimes L_i(\lambda))\check{R}(\lambda - \mu), \tag{2.4}$$

where $\check{R}$ is a $d^2 \times d^2$ matrix with normalization $\check{R}(0) = I$. The above tensor product lives in the space $V_1 \otimes V_2$ where $V_1 \simeq V_2 \simeq V$ is the d-dimensional space on which the $d \times d$ quantum matrix operators $L_i(\lambda)$ act. To indicate that $\check{R}$ acts on $V_1 \otimes V_2$, we rewrite it as $\check{R}_{V_1V_2}$ or $\check{R}_{12}$. Equation (2.4) implies as a compatibility condition the famous Yang-Baxter equation:

$$\check{R}_{23}(\lambda - \mu)\check{R}_{12}(\lambda)\check{R}_{23}(\mu) = \check{R}_{12}(\mu)\check{R}_{23}(\lambda)\check{R}_{12}(\lambda - \mu). \tag{2.5}$$

where $V_3 \simeq V$. It is usually written in terms of the variable R related to $\check{R}$ by

$$\check{R} = PR, \tag{2.6}$$

where P is a transposition operator, i.e.

$$P_{i,j}R_{jk}P_{i,j} = R_{ik}. \tag{2.7}$$

The relation (2.5) directly implies that the transfer matrix $T(\lambda)$

$$T(\lambda) = \mathrm{Tr}_{V_0} L_N(\lambda)...L_1(\lambda) \tag{2.8}$$

commutes with $T(\mu)$. Expanding the transfer matrix as

$$\ln T(\lambda) = 2i \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} Q_{n+1}, \tag{2.9}$$

one obtains a family of mutually commuting quantities $\{Q_n\}$:

$$[Q_n, Q_m] = 0. \tag{2.10}$$

The matrix elements of the operator $L_i(\lambda)$ are functions of the quantum variables $\mathbf{S}_i$, themselves $d \times d$ matrices. Matrix entries of L_i act then on a d-dimensional space, called the quantum space and denoted by V_0. Because $V_0 \simeq V$, we can set

$$L_i(\lambda) = R_{i0} \tag{2.11}$$

(up to a normalization factor). In that case, (2.4) becomes

$$R_{12}(\lambda - \mu)R_{10}(\lambda)R_{20}(\mu) = R_{20}(\mu)R_{10}(\lambda)R_{12}(\lambda - \mu), \tag{2.12}$$

which, thanks to (2.6), is the same as (2.5).

Spin models for which the Lax operator is the same as the R matrix are called fundamental [8]. In such cases, the quantum hamiltonian can be expressed directly in terms of the R matrix. Indeed, the transfer matrix is then a product of R matrices, one for each site:

$$T(\lambda) = \mathrm{Tr}_{V_0} R_{N0}(\lambda)...R_{10}(\lambda), \tag{2.13}$$

so that

$$\begin{aligned} \frac{dT(\lambda)}{d\lambda}|_{\lambda=0} &= \sum_i \mathrm{Tr}_{V_0} P_{N,0}...P_{i+1,0}\frac{dR_{i0}(\lambda)}{d\lambda}|_{\lambda=0}\ (P_{i,0}P_{i,0})P_{i-1,0}..P_{1,0} \\ &= \sum_i \frac{d\check{R}_{ii+1}(\lambda)}{d\lambda}|_{\lambda=0}\ \mathrm{Tr}_{V_0} P_{N,0}...P_{1,0} \end{aligned} \tag{2.14}$$

(in the first line we used the fact that $P_{i,0}^2 = I$). Up to a constant, this is simply the quantum chain hamiltonian

$$H = -T(0)^{-1}\frac{dT(\lambda)}{d\lambda}|_{\lambda=0} = -\sum_i \frac{dR_{ii+1}(\lambda)}{d\lambda}|_{\lambda=0}\ P_{i,i+1} = \sum_i h_{i,i+1}. \tag{2.15}$$

We next briefly review the relation between the Yang-Baxter equation and the boost operator [17]. The starting point is (2.12) with $1 \to k,\ 2 \to k+1$ and $\lambda = \nu + \mu$. The derivative of this expression with respect to ν evaluated at $\nu = 0$ yields

$$\frac{dR_{0k}(\mu)}{d\mu}R_{0k+1}(\mu) - R_{0k}\frac{dR_{0k+1}(\mu)}{d\mu} = [h_{k,k+1}, R_{0k}R_{0k+1}], \tag{2.16}$$

where $h_{k,k+1}$ is defined in (2.15). This result is then multiplied from the left by $\prod_{n<k} R_{0n}(\mu)$ and from the right by $\prod_{n>k+1} R_{0n}(\mu)$:

$$\begin{aligned} (\prod_{n<k} R_{0n}(\mu))\frac{dR_{0k}(\mu)}{d\mu}(\prod_{n>k} R_{0n}(\mu)) - (\prod_{n<k+1} R_{0n}(\mu))\frac{dR_{0k+1}(\mu)}{d\mu}(\prod_{n>k+1} R_{0n}(\mu)) \\ = [h_{k,k+1}, T(\mu)]. \end{aligned} \tag{2.17}$$

Multiplication by k followed by a summation over k from $-\infty$ to ∞ leads then to (1.8), with B defined by (1.7). From (2.9), this directly implies

$$[B, Q_n] = Q_{n+1}. \tag{2.18}$$

For finite chains with periodic boundary conditions, the argument is the same except that the coefficients are understood to be defined modulo N, the number of sites.

The logarithm in (2.9) ensures the locality of the charges Q_n, i.e. that Q_n does not contain interactions between spins at distances greater then n lattice units. As shown by Lüscher [9] for the XYZ model, these charges can be put in the form:

$$Q_n = \sum_{\{i_1,\ldots,i_{n-1}\}} G^T_{n-1}(i_1, \ldots, i_{n-1}), \tag{2.19}$$

where the summation is over ordered subsets $\{i_1, \ldots, i_{n-1}\}$ of the chain, and G^T is a translation covariant and totally symmetric function, with the locality property:

$$G^T_n(i_1, \ldots, i_n) = 0, \quad \text{for} \quad |i_n - i_1| \geq n. \tag{2.20}$$

For the infinite XXX chain, further properties of the conserved charges, including their completeness, have been proved in [34].[8]

In the rest of the paper we will reserve the notation $\{Q_n\}$ for the charges defined by (2.9). Note that a rescaling of the spectral parameter $\lambda \to \alpha^{-1}\lambda$ results in a multiplicative redefinition of the Q_n's : $Q_n \to \alpha^{n-1}Q_n$. For the XYZ model it is convenient to use the freedom to rescale the spectral parameter so that the logarithmic derivative of the transfer matrix becomes:

$$-T^{-1}(0)\dot{T}(\lambda)|_{\lambda=0} = \frac{1}{2i}Q_2 = \frac{1}{2i}\sum_{j\in\Lambda}[\lambda_x\sigma^x_j\sigma^x_{j+1} + \lambda_y\sigma^y_j\sigma^y_{j+1} + \lambda_z\sigma^z_j\sigma^z_{j+1}]. \tag{2.21}$$

With this normalization, the boost operator for the XYZ model reads:

$$B = \frac{1}{2i}\sum_{j\in\Lambda} j[\lambda_x\sigma^x_j\sigma^x_{j+1} + \lambda_y\sigma^y_j\sigma^y_{j+1} + \lambda_z\sigma^z_j\sigma^z_{j+1}]. \tag{2.22}$$

The structure of the conservation laws is however more transparent in another basis, which may be obtained by taking an appropriate linear combinations of Q_n's. Such a basis will be denoted $\{H_n\}$ and chosen so that $H_n = \frac{1}{(n-2)!}Q_n +$ a linear combination of lower order Q_i's.

[8] To our knowledge, no such completeness proof exists for the anisotropic case.

3. The conserved charges of the XYZ model: notation and generalities

3.1. Notation

We begin by introducing the necessary notation. We consider either a finite spin lattice with periodic boundary conditions ($\Lambda = \{1, \ldots, N\}$, with $N + 1 \equiv 1$) or an infinite one ($\Lambda = \mathbf{Z}$). Our constructions applies equally well to both cases, provided that addition in Λ is understood modulo N for the finite chain. We now introduce two general objects that enter naturally in the construction of the conserved charges: clusters and patterns.

A sequence of n lattice sites $\mathcal{C} = \{i_1, ..., i_n\}$, with $i_1 < i_2 < ... < i_n$, will be called a *cluster* of order n. A disordered cluster will refer to a sequence with the ordering condition relaxed.[9] A cluster starting at i_1 and ending at i_n, and containing non-adjacent sites has $k = i_n - i_1 + 1 - n$ *holes* (sites in between i_1 to i_n that are not included in $\mathcal{C}$); $k = 0$ for a cluster containing only contiguous spins. The set of all clusters of Λ of order n with k holes will be denoted as $\mathcal{C}^{(n,k)}$. For instance, $\mathcal{C}^{(4,2)}$ contains $\{1, 2, 3, 6\}$, $\{1, 2, 5, 6\}$, $\{1, 4, 5, 6\}$, $\{1, 2, 4, 6\}$, $\{1, 3, 4, 6\}$, $\{1, 3, 5, 6\}$ and all their translations.

Let $\mathcal{S}^{(n)}$ denote the set of all sequences of n basis spin matrices of the form

$$\mathcal{S} = \{\sigma_{i_1}^{a_1}, ..., \sigma_{i_n}^{a_n}\}, \tag{3.1}$$

where $a_i = x, y, z$, and $i_1 < i_2 < ... < i_n$. Clearly, such a sequence is completely specified by the cluster $\{i_1, \ldots, i_n\}$ and by the assignment of group indices, $\{a_1, \ldots, a_n\}$, which we call a *pattern*. We denote by $\mathcal{P}^{(n)}$ the set of all n-spin *patterns*, i.e. sequences of n elements taking values in the set $\{x, y, z\}$. There exist natural projections, assigning to a spin-sequence $\mathcal{S} \in \mathcal{S}^{(n)}$ its corresponding pattern or cluster:

$$p : \mathcal{S}^{(n)} \to \mathcal{P}^{(n)} \quad \text{and} \quad p' : \mathcal{S}^{(n)} \to \bigcup_k \mathcal{C}^{(n,k)} \tag{3.2}$$

given by:

$$p(\mathcal{S}) = \{a_1, a_2, \ldots, a_n\}, \qquad p'(\mathcal{S}) = \{i_1, i_2, \ldots, i_n\}. \tag{3.3}$$

We will write $\mathcal{C}^\pi$ to denote the sequence $\mathcal{S}$ with cluster C and pattern π.

[9] This definition of a cluster differs slightly from the one that we used previously in [31].

Next we construct the n-linear spin polynomials $\tilde{f}_n$, defined on basis sequences in $\mathcal{S}^{(n)}$, which will be the building blocks of our construction. First we define two different rescaling of the spin variables:[10]

$$\begin{aligned} \hat{\sigma}_j^a &= \sqrt{\lambda_a}\ \sigma_j^a, \\ \tilde{\sigma}_j^a &= \sqrt{\frac{\lambda_x\lambda_y\lambda_z}{\lambda_a}}\ \sigma_j^a. \end{aligned} \tag{3.4}$$

In these variables, the XYZ hamiltonian reads:

$$H_2 = \sum_{j\in\Lambda} \hat{\sigma}_j^a \hat{\sigma}_{j+1}^a. \tag{3.5}$$

Then, for any sequence of spins $\mathcal{S} \in \mathcal{S}^{(n)}$, we define $\mathbf{V}_m(\mathcal{S}) = \{V_m^x, V_m^y, V_m^z\}(\mathcal{S})$ (for $m < n$) as the vector product of the first m spins of the sequence, where the leftmost spin factor appears with a hat and all others with a tilde, with parentheses nested toward the left, i.e.:

$$\begin{aligned} &\mathbf{V}_1(\mathcal{S}) = \hat{\boldsymbol{\sigma}}_{i_1}, \\ &\mathbf{V}_2(\mathcal{S}) = \hat{\boldsymbol{\sigma}}_{i_1} \times \tilde{\boldsymbol{\sigma}}_{i_2}, \\ &\mathbf{V}_3(\mathcal{S}) = (\hat{\boldsymbol{\sigma}}_{i_1} \times \tilde{\boldsymbol{\sigma}}_{i_2}) \times \tilde{\boldsymbol{\sigma}}_{i_3}, \\ &\dots \\ &\mathbf{V}_m(\mathcal{S}) = \mathbf{V}_{m-1}(\mathcal{S}) \times \tilde{\boldsymbol{\sigma}}_{i_m}. \end{aligned} \tag{3.6}$$

We also denote by $a_m(\pi)$ the " direction" of $\mathbf{V}_m(\mathcal{C}^\pi)$, i.e:

$$a_m(\pi) = (\dots(a_1 \times a_2) \times a_3)\dots \times a_m). \tag{3.7}$$

Then, we construct scalar n-linear polynomials, defined in the space of spin sequences, from the scalar product of $\mathbf{V}_{n-1}(\mathcal{S})$ and $\hat{\boldsymbol{\sigma}}_{i_n}$, multiplied by an appropriate constant $g(\mathcal{S})$:

$$\tilde{f}_n(\mathcal{S}) = g(\mathcal{S})(\mathbf{V}_{n-1}(\mathcal{S})\cdot\hat{\boldsymbol{\sigma}}_{i_n}), \tag{3.8}$$

with $\tilde{f}_0(\mathcal{S}) = \tilde{f}_1(\mathcal{S}) = 0$. The coefficient $g(\mathcal{S})$ is a polynomial in $\lambda_x, \lambda_y, \lambda_z$, determined by the pattern and the positions of the holes in the cluster:

[10] Although these rescalings involve square roots of the coupling constants, the resulting expressions for charges will contain only integer powers of the couplings.

(i)- If $\mathcal{S}$ contains only contiguous spins, $g(\mathcal{S}) = 1$. In this case,

$$\begin{aligned}\tilde{f}_n(\mathcal{S}) &= ((\ldots((\hat{\sigma}^{a_1}_{i_1} \times \tilde{\sigma}^{a_2}_{i_2}) \times \tilde{\sigma}^{a_3}_{i_3}) \times \ldots)\tilde{\sigma}^{a_{n-1}}_{i_{n-1}}) \cdot \hat{\sigma}^{a_n}_{i_n}, \\ &= \epsilon(\pi) \quad \hat{\sigma}^{a_1}_{i_1}\tilde{\sigma}^{a_2}_{i_2} \ldots \tilde{\sigma}^{a_3}_{i_3}\tilde{\sigma}^{a_{n-1}}_{i_{n-1}}\hat{\sigma}^{a_n}_{i_n},\end{aligned} \tag{3.9}$$

where

$$\epsilon(\pi) = \mathrm{sgn}[a_{n-1}(\pi)]\delta_{a_{n-1}(\pi),a_n} \tag{3.10}$$

with the convention $\mathrm{sgn}(0) = 0$. Hence, $\epsilon(\pi)$ can be either 0 or ± 1. For example, we have:

$$\begin{aligned}&\tilde{f}_2(\{\sigma^x_i, \sigma^x_{i+1}\}) = \lambda_x\ \sigma^x_i\sigma^x_{i+1}, \\ &\tilde{f}_3(\{\sigma^x_i, \sigma^y_{i+1}, \sigma^z_{i+2}\}) = \lambda_x\lambda_y\ \sigma^x_i\sigma^y_{i+1}\sigma^z_{i+2}, \\ &\tilde{f}_3(\{\sigma^x_i, \sigma^y_{i+1}, \sigma^y_{i+2}\}) = 0, \\ &\tilde{f}_4(\{\sigma^x_i, \sigma^y_{i+1}, \sigma^y_{i+2}, \sigma^x_{i+3}\}) = -{\lambda_x}^2\lambda_z\ \sigma^x_i\sigma^y_{i+1}\sigma^y_{i+2}\sigma^x_{i+3}.\end{aligned} \tag{3.11}$$

(ii)- For clusters containing holes, the coefficient $g(\mathcal{S})$ is given by

$$g(\mathcal{S}) = \prod_{\substack{j=i_1+1; \\ j\notin\mathcal{C}}}^{i_{n-1}} \frac{\lambda_x\lambda_y\lambda_z}{\lambda_{|a_{L(j)}(\pi)|}}, \tag{3.12}$$

where $L(j)$ is the number of sites in $\mathcal{C}$ to the left of site j and $a_{L(j)}(\pi)$ has been defined above in (3.7). For example,

$$\begin{aligned}&\tilde{f}_2(\{\sigma^x_i, \sigma^x_{i+2}\}) = (\lambda_y\lambda_z)\lambda_x\ \sigma^x_i\sigma^x_{i+2}, \\ &\tilde{f}_3(\{\sigma^x_i, \sigma^y_{i+2}, \sigma^z_{i+3}\}) = (\lambda_y\lambda_z)\lambda_x\lambda_y\ \sigma^x_i\sigma^y_{i+2}\sigma^z_{i+3}, \\ &\tilde{f}_3(\{\sigma^x_i, \sigma^y_{i+1}, \sigma^y_{i+3}, \sigma^x_{i+4}\}) = -(\lambda_x\lambda_y){\lambda_x}^2\lambda_z\ \sigma^x_i\sigma^y_j\sigma^y_k\sigma^x_l,\end{aligned} \tag{3.13}$$

(where the coefficient $g(\mathcal{S})$ is indicated in parentheses).

Finally, we define the linear spaces $\mathcal{F}_{n,k}$ spanned by the translation invariant local polynomials built from $\tilde{f}_n$'s:[11]

$$\mathcal{F}_{n,k} = \sum_{\pi\in\mathcal{P}^{(n)}} \alpha(\pi) \sum_{\mathcal{C}\in\mathcal{C}^{n,k}} \tilde{f}_n(\mathcal{C}^\pi), \tag{3.14}$$

where $\alpha(\pi)$ are some real-valued coefficients.

[11] $\mathcal{F}_{n,k}$ is a linear subspace of dimension $3\cdot 2^{n-2}$ of the space of observables; since the product of two local polynomials is not local, it is not an algebra.

In the isotropic case, i.e. for the XXX model, all the coupling dependent coefficients become one, which results in a significant simplification. We can then define the polynomials f_n whose arguments are clusters in $\mathcal{C}^{n,k}$, as follows:

$$f_n(\mathcal{C}) = \sum_{\pi \in \mathcal{P}^{(n)}} \tilde{f}_n(\mathcal{C}^\pi). \tag{3.15}$$

In particular, omitting the argument of the f_n's for brevity, we have

$$\begin{aligned} f_0 &= f_1 = 0, \\ f_2 &= \boldsymbol{\sigma}_{i_1} \cdot \boldsymbol{\sigma}_{i_2}, \\ f_3 &= (\boldsymbol{\sigma}_{i_1} \times \boldsymbol{\sigma}_{i_2}) \cdot \boldsymbol{\sigma}_{i_3}, \\ f_4 &= ((\boldsymbol{\sigma}_{i_1} \times \boldsymbol{\sigma}_{i_2}) \times \boldsymbol{\sigma}_{i_3}) \cdot \boldsymbol{\sigma}_{i_4}. \end{aligned} \tag{3.16}$$

The multilinear polynomials f_n's and $\tilde{f}_n$'s satisfy an interesting property which is that the dot product can be placed at an arbitrary position, provided that parentheses to its left (right) are nested toward the left (right), e.g:

$$\begin{aligned} f_5 &= (((\boldsymbol{\sigma}_{i_1} \times \boldsymbol{\sigma}_{i_2}) \times \boldsymbol{\sigma}_{i_3}) \times \boldsymbol{\sigma}_{i_4}) \cdot \boldsymbol{\sigma}_{i_5} = ((\boldsymbol{\sigma}_{i_1} \times \boldsymbol{\sigma}_{i_2}) \times \boldsymbol{\sigma}_{i_3}) \cdot (\boldsymbol{\sigma}_{i_4} \times \boldsymbol{\sigma}_{i_5}) \\ &= \boldsymbol{\sigma}_{i_1} \cdot (\boldsymbol{\sigma}_{i_2} \times (\boldsymbol{\sigma}_{i_3} \times (\boldsymbol{\sigma}_{i_4} \times \boldsymbol{\sigma}_{i_5}))). \end{aligned} \tag{3.17}$$

This is a direct consequence of the familiar vector identity:

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}). \tag{3.18}$$

3.2. The structure of the XYZ conservation laws

The conserved charges of the XYZ model have the general form:

$$Q_n = \sum_{k=0}^{[n/2]-1} \sum_{\ell=0}^{k} \tilde{F}^n_{n-2k,\ell}, \tag{3.19}$$

where the square bracket stands for integer part, and $\tilde{F}^n_{n-2k,\ell}$ is an element of $\mathcal{F}_{n-2k,\ell}$. More precisely,

$$\tilde{F}^n_{n-2k,\ell} = \sum_{\pi \in \mathcal{P}^{(n-2k)}} \theta^n_{n-2k,\ell}(\pi) \sum_{\mathcal{C} \in \mathcal{C}^{(n-2k,\ell)}} \tilde{f}_{n-2k}(\mathcal{C}^\pi), \tag{3.20}$$

where $\theta^n_{n-2k,\ell}$ is a polynomial of degree $2(k-l)$ in the coupling constants $\lambda_x, \lambda_y, \lambda_z$. It is convenient to think of Q_n in terms of a triangle, consisting of the operators $\tilde{F}^n_{n-2k,\ell}$:

$$\begin{array}{ccccccc} & & & & & \tilde{F}^n_{n,0} & \\ & & & & \tilde{F}^n_{n-2,1} & \tilde{F}^n_{n-2,0} & \\ & & & \tilde{F}^n_{n-4,2} & \tilde{F}^n_{n-4,1} & \tilde{F}^n_{n-4,0} & \\ & & \tilde{F}^n_{n-6,3} & \tilde{F}^n_{n-6,2} & \tilde{F}^n_{n-6,1} & \tilde{F}^n_{n-6,0} & \\ & \tilde{F}^n_{n-8,4} & \tilde{F}^n_{n-8,3} & \tilde{F}^n_{n-8,2} & \tilde{F}^n_{n-8,1} & \tilde{F}^n_{n-8,0} & \\ & & & \ldots & & & \end{array} \tag{3.21}$$

The bottom edge of the triangle is formed by the sequence $\{\tilde{F}^n_{2,k}\}_{k=0,\ldots,[n/2]-1}$ if n is even, or $\{\tilde{F}^n_{3,k}\}_{k=0,\ldots,[n/2]-1}$ if n is odd. Note that linear combinations of the form

$$H_n = \sum_{k=0}^{[n/2]-1} p_{n-2k}(\lambda_x, \lambda_y, \lambda_z) Q_{n-2k}, \tag{3.22}$$

where p_{n-2k} is an arbitrary polynomial of degree $n-2k$ in the coupling constants preserve the form of the triangle, redefining only the polynomials $\theta(\pi)$.

The problem is how to determine the coefficients $\theta(\pi)$. In section (5), where we prove (3.19), we give a recursive method to calculate $\tilde{F}^n_{n-2k,\ell}$. Some of the coefficients are easy to calculate: e.g. one finds that $\theta^n_{n-2k,k}(\pi) = (n-2)!$ for all patterns π (thus $\tilde{F}^n_{n-2k,k} \sim \sum_{\pi\in\mathcal{P}^{(n-2k)}} \sum_{\mathcal{C}\in\mathcal{C}^{(n-2k,k)}} \tilde{f}_n(\mathcal{C}^\pi)$). This fixes the coefficients on the left edge of the triangle. Unfortunately, finding all the coefficients in the general anisotropic XYZ case is a very difficult task and we have not succeeded in disentangling the equations for the θ's apart for those on the left edge. However, there are two particular cases in which all the coefficients are relatively simple and can be determined exactly, namely when $\lambda_z = 0$ or in the isotropic limit where $\lambda_x = \lambda_y = \lambda_z$. In the XY model, which is analyzed in section (6), the triangle degenerates into its right edge. On the other hand, for the XXX model, treated in detail in the next section, all the coupling-dependent coefficients become constants. In that case, by an appropriate transformation of the type (3.22), one can eliminate all the terms on the right edge, except for $\tilde{F}^n_{n,0}$. This decoupling provides enough simplification in the recursion relations for the coefficients to allow for an exact solution.

For the XXZ model, the recursion formulae are somewhat simplified, being dependent only on a single parameter λ_z/λ_x. However, this simplification is not as dramatic as in the XXX case, and it does not seem to permit a solution in closed form for all the coefficients.

4. The explicit form of the conservation laws in the XXX case

4.1. The general formula

For the isotropic XXX model the only quantities that enter in the construction of the local integrals of motion are

$$F_{n,k} = \sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} f_n(\mathcal{C}). \tag{4.1}$$

Quite remarkably, the linear combination of $F_{n,k}$'s that builds up H_n is encoded in a simple tree, in which the vertices are labeled by the $F_{n,k}$'s. This tree is displayed in Fig. 1.

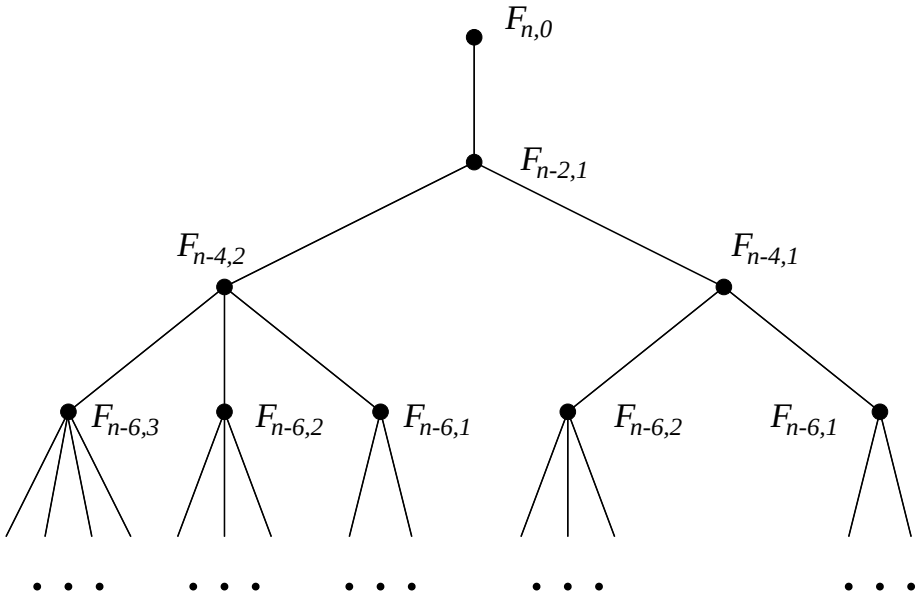

Fig. 1. The tree structure corresponding to H_n. The tree stops with the terms $F_{2,\ell}$ ($F_{3,\ell}$) when n is even (odd). Note that the trees corresponding to H_{2m} and H_{2m+1} have identical structure, differing only in labeling of their vertices.

The sum of all the vertices of the tree, with each vertex contributing with unit weight, gives H_n. Summing up all the vertices of the H_n tree leads then to the following expression:

$$H_n = F_{n,0} + \sum_{k=1}^{[n/2]-1} \sum_{\ell=1}^{k} \alpha_{k,\ell} F_{n-2k,\ell}, \tag{4.2}$$

where the coefficients $\alpha_{k,\ell}$ are defined recursively by the relation:

$$\alpha_{k+1,\ell} = \sum_{m=\ell-1}^{k} \alpha_{k,m}, \tag{4.3}$$

with $\alpha_{1,1} = 1$ and $\alpha_{k,0} = 0$. Notice that $\alpha_{k,1} = \alpha_{k,2}$ for $k \geq 2$. These coefficients turn out to be generalized Catalan numbers, as can be seen by rewriting the recurrence relation (4.3) in the form:

$$\alpha_{k,\ell} = \alpha_{k-1,\ell-1} + \alpha_{k,\ell+1}, \tag{4.4}$$

with the convention $\alpha_{k,\ell} = 0$ if $\ell > k$. This is the defining relation for the generalized Catalan numbers, $\alpha_{k,\ell} = C_{2k-l-1,\ell}$, with $C_{n,m}$ given by

$$C_{n,m} = \binom{n-1}{p} - \binom{n-1}{p-2}, \tag{4.5}$$

where $\binom{a}{b}$ are the binomial coefficients, with $p = [(n-m+1)/2]$, $m+n$ odd and $m < n+2$. In particular, $\alpha_{k,1} = C_{2k,1}$ reproduce the familiar sequence of Catalan numbers: $\{1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862 \ldots\}$. The tree in Fig. 1 is known as a Catalan tree.

The first few conserved charges are:

$$\begin{aligned}
H_2 &= F_{2,0}, \\
H_3 &= F_{3,0}, \\
H_4 &= F_{4,0} + F_{2,1}, \\
H_5 &= F_{5,0} + F_{3,1}, \\
H_6 &= F_{6,0} + F_{4,1} + F_{2,2} + F_{2,1}, \\
H_7 &= F_{7,0} + F_{5,1} + F_{3,2} + F_{3,1}, \\
H_8 &= F_{8,0} + F_{6,1} + F_{4,2} + F_{4,1} + F_{2,3} + 2F_{2,2} + 2F_{2,1}, \\
H_9 &= F_{9,0} + F_{7,1} + F_{5,1} + F_{5,2} + F_{3,3} + 2F_{3,2} + 2F_{3,1}, \\
H_{10} &= F_{10,0} + F_{8,1} + F_{6,2} + F_{6,1} + F_{4,3} + 2F_{4,2} + 2F_{4,1} \\
&\quad + F_{2,4} + 3F_{2,3} + 5F_{2,2} + 5F_{2,1}.
\end{aligned} \tag{4.6}$$

We note that (4.2) can be put in the form (3.19) with $F^n_{n,0} = F_{n,0}$, $F^n_{n-2k,0} = 0$ for $0 < k < [n/2] - 1$, $F^n_{n-2k,\ell} = \alpha_{k,\ell} F_{n-2k,l}$.

The above construction gives $N-1$ charges $\{H_2, \ldots H_N\}$ for the XXX chain of length N with periodic boundary conditions. We can add to the set $\{H_n\}$ any of the three

components of the total spin, $H_1^a = \sum_{j\in\Lambda} S_j^a$. As the $F_{n,k}$'s are invariant under global spin rotation, $[H_1^a, H_n] = 0$.[12] For the infinite isotropic Heisenberg chain we have then an infinite sequence of independent local charges $\{H_1^a, H_2, ..., H_n, ...\}$.

4.2. Details of the proof

We now present the direct algebraic proof of the mutual commutativity of all the $\{H_n\}$'s. We will first show that $[H_2, H_n] = 0$, by evaluating the commutator:

$$[H_2, f_n(\mathcal{C})] = \sum_{j\in\Lambda} [\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}, f_n(\mathcal{C})]. \tag{4.7}$$

The proof is rather lengthy and some of its steps are quite tedious. It is divided into nine parts.

a) Three useful identities

To proceed, we need a few simple identities for commutators in the $su(2)$ tensor product space. Let $\mathbf{A} = \{\sigma_j^x, \sigma_j^y, \sigma_j^z\}$ and $\mathbf{B} = \{\sigma_k^x, \sigma_k^y, \sigma_k^z\}$ denote the basis of two different $(j \neq k)$ $su(2)$ factors, and let $\mathbf{L} = \{L^1, L^2, L^3\}$ and $\mathbf{R} = \{R^1, R^2, R^3\}$ be arbitrary tensor products commuting with all components of both $\mathbf{A}$ and $\mathbf{B}$. The following identities hold:

$$[\mathbf{A}\cdot\mathbf{B}, (\mathbf{B}\times\mathbf{L})\cdot\mathbf{R}] = -2i((\mathbf{A}\times\mathbf{B})\times\mathbf{L})\cdot\mathbf{R}, \tag{4.8}$$

$$[\mathbf{A}\cdot\mathbf{B}, (\mathbf{A}\times\mathbf{B})\cdot\mathbf{R}] = 4i(\mathbf{B}\cdot\mathbf{R}) - 4i(\mathbf{A}\cdot\mathbf{R}), \tag{4.9}$$

$$[\mathbf{A}\cdot\mathbf{B}, ((\mathbf{L}\times\mathbf{A})\times\mathbf{B})\cdot\mathbf{R}] = 2i(\mathbf{L}\times\mathbf{B})\cdot\mathbf{R} - 2i(\mathbf{L}\times\mathbf{A})\cdot\mathbf{R}. \tag{4.10}$$

These identities are simple consequences of the basic $su(2)$ multiplication rule for the components of $\mathbf{A}, \mathbf{B}$:

$$A^a A^b = \delta^{ab} + i\epsilon_{abc} A^c. \tag{4.11}$$

[12] Thus the charges (4.2) survive in the presence of a magnetic field coupled to any of the components of the total spin.

b) Commuting $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}$ with $f_n(\mathcal{C})$

The above identities are helpful in evaluating the commutators of the basic links $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}$ with $f_n(\mathcal{C})$. It is remarkable that all such commutators are of the form $f_{n\pm1}(\mathcal{C}')$, where the cluster $\mathcal{C}'$ can be obtained by applying a number of simple transformations to the cluster $\mathcal{C} = \{i_1, \ldots, i_n\}$:

$$\begin{aligned} {}_{i_1-1}\mathcal{C} &\equiv \{i_1 - 1, i_1, \ldots, i_n\}, \\ \mathcal{C}_{i_n+1} &\equiv \{i_1, \ldots, i_n, i_n + 1\}, \\ \mathcal{C}_{\widehat{i_k}} &\equiv \{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n\}, \\ \mathcal{C}_{i_j \to i_k i_\ell} &\equiv \{i_1, \ldots, i_{j-1}, (i_k, i_\ell), i_{j+1}, \ldots, i_n\}. \end{aligned} \tag{4.12}$$

Note that the last operation is defined only if i_k, i_ℓ are not in $\mathcal{C}_{\widehat{i_j}}$; the brackets indicate the change in nesting of vector products when the expression appears as an argument of the polynomial f_{n+1}. As a result, this fourth operation produces a disordered cluster, unless i_j is at the extremity, i.e. $j = 1$ or $j = n$. Although this may not be completely clear at this point, it will be illustrated shortly by an example. On the other hand, the first three transformations always yield ordered clusters.

For $\mathcal{C} \in \mathcal{C}^{(n,k)}$ the terms arising in (4.7) correspond to clusters with $(n', k') = (n + 1, k),\ (n + 1, k - 1), (n - 1, k),\ (n - 1, k - 1)$. More precisely, using the above identities, the different terms in (4.7) are found to be:

$$\begin{aligned} [\boldsymbol{\sigma}_{i_1-1} \cdot \boldsymbol{\sigma}_{i_1}, f_n(\mathcal{C})] &= -2i f_{n+1}({}_{i_1-1}\mathcal{C}), \\ [\boldsymbol{\sigma}_{i_n-1} \cdot \boldsymbol{\sigma}_{i_n}, f_n(\mathcal{C})] &= 2i f_{n+1}(\mathcal{C}_{i_n+1}), \\ [\boldsymbol{\sigma}_{i_1} \cdot \boldsymbol{\sigma}_{i_1+1}, f_n(\mathcal{C})] &= 4i f_{n-1}(\mathcal{C}_{\widehat{i_1}}) - 4i f_{n-1}(\mathcal{C}_{\widehat{i_1+1}}) \quad \text{if} \quad i_2 = i_1 + 1 \\ &= 2i f_{n+1}({}_{i_1 i_1+1}\mathcal{C}_{\widehat{i_1}}) \quad \text{if} \quad i_2 \neq i_1 + 1, \\ [\boldsymbol{\sigma}_{i_n-1} \cdot \boldsymbol{\sigma}_{i_n}, f_n(\mathcal{C})] &= 4i f_{n-1}(\mathcal{C}_{\widehat{i_n-1}}) - 4i f_{n-1}(\mathcal{C}_{\widehat{i_n}}) \quad \text{if} \quad i_{n-1} = i_n - 1 \\ &= -2i f_{n+1}(\mathcal{C}_{\widehat{i_n} i_n-1 i_n}) \quad \text{if} \quad i_{n-1} \neq i_n - 1, \\ [\boldsymbol{\sigma}_{i_j} \cdot \boldsymbol{\sigma}_{i_j+1}, f_n(\mathcal{C})] &= 2i f_{n-1}(\mathcal{C}_{\widehat{i_j}}) - 2i f_{n-1}(\mathcal{C}_{\widehat{i_j+1}}) \quad \text{if} \quad i_j, i_{j+1} \in \mathcal{C} \\ &= -2i f_{n+1}(\mathcal{C}_{i_j \to i_j i_j+1}) \quad \text{if} \quad i_j \in \mathcal{C},\ , i_{j+1} \notin \mathcal{C}, \end{aligned} \tag{4.13}$$

where in the last case, it is assumed that $j \neq 1, n - 1$.

The last term in 4.13 contains contributions from disordered clusters. For example, consider the cluster $\mathcal{C} = \{1, 2, 4, 5\}$ and the commutator of

$$f_4(\mathcal{C}) = \boldsymbol{\sigma}_1 \cdot (\boldsymbol{\sigma}_2 \times (\boldsymbol{\sigma}_4 \times \boldsymbol{\sigma}_5)) = (\boldsymbol{\sigma}_2 \times (\boldsymbol{\sigma}_4 \times \boldsymbol{\sigma}_5)) \cdot \boldsymbol{\sigma}_1. \tag{4.14}$$

with $\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_3$. Taking for f_4 the second expression above allows us to use (4.8) directly, with the result

$$[\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_3, (\boldsymbol{\sigma}_2 \times (\boldsymbol{\sigma}_4 \times \boldsymbol{\sigma}_5)) \cdot \boldsymbol{\sigma}_1] = -2i((\boldsymbol{\sigma}_3 \times \boldsymbol{\sigma}_2) \times (\boldsymbol{\sigma}_4 \times \boldsymbol{\sigma}_5)) \cdot \boldsymbol{\sigma}_1 \tag{4.15}$$

We see clearly that the nesting of the spins has been modified (that is the parentheses cannot be all nested toward one side or the other). In other words, the result cannot be written in terms of $f_5(\{1,2,3,4,5\})$. As this calculation illustrates, the special property (3.18) makes the direct application of the identities (4.8) -(4.10) straightforward.

c) Summation over links inside a cluster

Adding up all these terms gives:

$$[H_2, f_n(\mathcal{C})] = a_{n+1,k}(\mathcal{C}) + b_{n-1,k+1}(\mathcal{C}) + d_{n+1,k-1}(\mathcal{C}) + e_{n-1,k}(\mathcal{C}) + r(\mathcal{C}), \tag{4.16}$$

where the various quantities appearing in this expression are defined by:

$$\begin{aligned}
a_{n+1,k}(\mathcal{C}) &= -2if_{n+1}({}_{i_1-1}\mathcal{C}) + 2if_{n+1}(\mathcal{C}_{i_n+1}), \\
b_{n-1,k+1}(\mathcal{C}) &= -4if_{n-1}(\mathcal{C}_{\widehat{i_2}})\delta_{i_1+1,i_2} + 4if_{n-1}(\mathcal{C}_{\widehat{i_{n-1}}})\delta_{i_{n-1}+1,i_n} \\
&\quad + 2i\sum_{j=2}^{n-2}[f_{n-1}(\mathcal{C}_{\widehat{i_j}}) - f_{n-1}(\mathcal{C}_{\widehat{i_{j+1}}})]\delta_{i_{j+1},i_j+1}, \\
d_{n+1,k-1}(\mathcal{C}) &= 2if_{n+1}({}_{i_1\,i_1+1}\mathcal{C}_{\widehat{i_1}})(1-\delta_{i_1,i_2-1}) \\
&\quad - 2if_{n+1}(\mathcal{C}_{\widehat{i_n}\,i_{n-1}i_n})(1-\delta_{i_n,i_{n-1}+1}), \\
e_{n-1,k}(\mathcal{C}) &= 4if_{n-1}(\mathcal{C}_{\widehat{i_1}})\delta_{i_1+1,i_2} - 4if_{n-1}(\mathcal{C}_{\widehat{i_n}})\delta_{i_{n-1}+1,i_n}, \\
r(\mathcal{C}) &= -2i\sum_{j=2}^{n-2} f_{n+1}(\mathcal{C}_{i_j\to i_j i_j+1})(1-\delta_{i_{j+1},i_j+1}). \\
&\quad + 2i\sum_{j=3}^{n-1} f_{n+1}(\mathcal{C}_{i_j\to i_j-1 i_j})(1-\delta_{i_{j-1},i_j-1})
\end{aligned} \tag{4.17}$$

Caution: in the above formulae, one should distinguish carefully between sites $i_j + 1$ and i_{j+1}.

d) Summation over clusters

With the exception of $r(\mathcal{C})$, all of the terms on the right hand-side of (4.16) involve only regular nested product of the full set of spin variables. Fortunately, when summing up over all clusters in $\mathcal{C}^{(n,k)}$, the unwanted contributions from improperly-nested spin sequences cancel:

$$\sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} r(\mathcal{C}) = 0. \tag{4.18}$$

This can be seen as follows. r-type terms arise only from clusters containing holes. Consider a particular cluster $\mathcal{C}'$ having a hole at site j but not at site $j+1$. Let us write its associated polynomial f as

$$f(\mathcal{C}') = (\mathbf{L}\times\boldsymbol{\sigma}_{j+1})\cdot\mathbf{R}, \tag{4.19}$$

where $\mathbf{L}$ stands for the contribution of all the spins at the left of the site j and $\mathbf{R}$ for that of the spins at the right of $j+1$. The contribution of this cluster to the commutator with $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_{j+1}$ is

$$[\boldsymbol{\sigma}_j\boldsymbol{\sigma}_{j+1}, (\mathbf{L}\times\boldsymbol{\sigma}_{j+1})\cdot\mathbf{R}] = -2i(\mathbf{L}\times(\boldsymbol{\sigma}_j\times\boldsymbol{\sigma}_{j+1}))\cdot\mathbf{R}. \tag{4.20}$$

In the summation over all possible clusters having the same number of spins and same number of holes, we will encounter another cluster $\mathcal{C}''$, which differs from $\mathcal{C}'$ only in that the hole which was in position j now appears in position $j+1$. The f polynomial of this cluster being

$$f(\mathcal{C}'') = (\mathbf{L}\times\boldsymbol{\sigma}_j)\cdot\mathbf{R}, \tag{4.21}$$

its contribution to the commutator with $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_{j+1}$ is

$$[\boldsymbol{\sigma}_j\boldsymbol{\sigma}_{j+1}, (\mathbf{L}\times\boldsymbol{\sigma}_j)\cdot\mathbf{R}] = -2i(\mathbf{L}\times(\boldsymbol{\sigma}_{j+1}\times\boldsymbol{\sigma}_j))\cdot\mathbf{R}, \tag{4.22}$$

which is exactly the negative of (4.20). Hence, in the summation over all clusters of a fixed type, all the contributing r-terms cancel two by two.

It then follows that the commutator $[H_2, F_{n,k}]$ can be expressed as a linear combination of terms of the type $F_{n',k'}$. Equivalently, we may interpret this commutator as a transformation of the set of all clusters into itself. Now, with an argument similar to the one used to show the cancellation of the r terms, we can prove that the sum over the e-type terms also adds up to zero:

$$\sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} e_{n-1,k}(\mathcal{C}) = 0. \tag{4.23}$$

Let us illustrate this calculation by a detailed example. Take $N = 7,\ n = 4$ and $k = 1$. The different clusters $\mathcal{C}^{(4,1)}$ are

$$\begin{aligned}\{\mathcal{C}^{(4,1)}\}_{N=7} =&\{\{1,2,4,5\},\{2,3,5,6\},\{3,4,6,7\},\{1,3,4,5\},\{2,4,5,6\},\\ &\{3,5,6,7\},\{1,2,3,5\},\{2,3,4,6\},\{3,4,5,7\}\}.\end{aligned} \tag{4.24}$$

The $e_{3,1}$ terms for the first three clusters are

$$\begin{aligned}4i[f_3(\{2,4,5\})-f_3(\{1,2,4\})+f_3(\{3,5,6\})-f_3(\{2,3,5\})\\ +f_3(\{4,6,7\})-f_3(\{3,4,6\})],\end{aligned} \tag{4.25}$$

those for the following three clusters are

$$4i[-f_3(\{1,3,4\})-f_3(\{2,4,5\})-f_3(\{3,5,6\})], \tag{4.26}$$

and for the last three, these are

$$4i[f_3(\{2,3,5\})+f_3(\{3,4,6\})+f_3(\{4,5,7\})]. \tag{4.27}$$

Adding up all these contributions, we see that terms cancel in pairs.

Therefore,

$$[H_2, F_{n,k}] = A_{n+1,k} + D_{n+1,k-1} + B_{n-1,k+1}, \tag{4.28}$$

where

$$\begin{aligned}A_{n+1,k} &= \sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} a_{n+1,k}(\mathcal{C}),\\ B_{n-1,k+1} &= \sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} b_{n-1,k+1}(\mathcal{C}),\\ D_{n+1,k-1} &= \sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} d_{n+1,k+1}(\mathcal{C}).\end{aligned} \tag{4.29}$$

e) The special case where $n = N$

In the case where $n = N$, the calculation of (4.7) is slightly different, as a new type of term appears in the commutator of the hamiltonian and the highest order term in H_N, i.e.

$$F_{N,0} = \sum_{j=1}^{N} f_N(\mathcal{C}_j), \tag{4.30}$$

where

$$\mathcal{C}_j = \{j, j+1\ldots, j+N-1\}. \tag{4.31}$$

This commutator reads:

$$[H_2, f_N(\mathcal{C}_j)] = g_j + e_{N-1,1}(\mathcal{C}_j) + b_{N-1,1}(\mathcal{C}_j), \tag{4.32}$$

where

$$\begin{aligned} g_j &= [\boldsymbol{\sigma}_{j+N-1} \cdot \boldsymbol{\sigma}_j, f_N(\mathcal{C}_j)] \\ &= 2i\epsilon^{ka_0b_0}\epsilon^{b_0a_1b_2}\ldots\epsilon^{b_{N-3}a_{N-2}k}(\sigma_{j+1}^{a_0}\sigma_{j+2}^{a_1}\ldots\sigma_{j+N-1}^{a_{N-2}} - \sigma_j^{a_0}\sigma_{j+1}^{a_1}\ldots\sigma_{j+N-2}^{a_{N-2}}). \end{aligned} \tag{4.33}$$

It is obvious that

$$\sum_{j=1}^{N} g_j = 0. \tag{4.34}$$

Summing up over all $\mathcal{C}_j$ in (4.32) gives then

$$[H_2, F_{N,0}] = B_{N-1,1}. \tag{4.35}$$

f) Conditions for the commutativity of H_n with H_2

A simple consequence of (4.28) and (4.35), is that the commutation with the hamiltonian changes the order of the clusters in $F_{n,k}$ from n to $n \pm 1$. Therefore, to prove that the charge H_n of the form (4.2) commutes with the hamiltonian, it is sufficient to show, for any $n \leq N$, that:
(i) $[H_2, F_{n,0}]$ does not contain terms of order $n+1$,
(ii) $[H_2, F_{n,k} + \sum_{\ell=1}^{k+1} F_{n-2,\ell}]$ does not contain terms of order $n-1$.
The first assertion is easily verified for $n < N$ since $A_{n+1,0} = 0$, by translational symmetry, that is

$$A_{n+1,0} = 2i\sum_{j\in\Lambda}[-f_{n+1}(\{j-1, j, ..., j+n\}) + f_{n+1}(\{j, j+1, ..., j+n+1\})] = 0. \tag{4.36}$$

For $n = N$, (i) immediately follows from (4.35). The second assertion is more difficult to prove. It is equivalent to:

$$B_{n-1,k+1} + \sum_{\ell=1}^{k+1}(A_{n-1,\ell} + D_{n-1,\ell-1}) = 0. \tag{4.37}$$

As the sum above contains terms corresponding to clusters of order $n-1$, with hole numbers ranging from 0 to $k+1$, it can vanish if and only if all terms with a given number of holes cancel, i.e.

$$B_{n-1,k+1} + A_{n-1,k+1} = 0, \tag{4.38}$$

$$A_{n-1,\ell} + D_{n-1,\ell} = 0, \quad (1 \leq \ell \leq k) \tag{4.39}$$

$$D_{n-1,0} = 0. \tag{4.40}$$

(4.39) and (4.40) are direct consequences of (4.17). In fact, they follow immediately once we notice that $D_{n-1,\ell}$ can be rewritten as

$$D_{n-1,\ell} = -\sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} a_{n+1,\ell}(\mathcal{C}) \tag{4.41}$$

The proof of (4.38) is much more tedious.

g) A resummation trick

In order to demonstrate (4.38), one first has to rewrite the B term in a way that can be compared with the form of A. B naturally decomposes into two parts;

$$B_{n-1,k+1} = B^{(1)}_{n-1,k+1} + B^{(2)}_{n-1,k+1}, \tag{4.42}$$

with

$$B^{(1)}_{n-1,k+1} = -4i \sum_{\mathcal{C}\in\mathcal{C}^{(n-1,k+1)}} [f_{n-1}(\mathcal{C}_{\widehat{i_2}})\delta_{i_1+1,i_2} - f_{n-1}(\mathcal{C}_{\widehat{i_{n-1}}})\delta_{i_{n-1}+1,i_n}], \tag{4.43}$$

and

$$B^{(2)}_{n-1,k+1} = 2i \sum_{\mathcal{C}\in\mathcal{C}^{(n-1,k+1)}} \sum_{j=2}^{n-2} [f_{n-1}(\mathcal{C}_{\widehat{i_j}}) - f_{n-1}(\mathcal{C}_{\widehat{i_{j+1}}})]\delta_{i_{j+1},i_j+1}. \tag{4.44}$$

Both parts vanish identically for $n = 2, 3$. For $n \geq 4$, the set of clusters $\mathcal{C}^{(n-1,k+1)}$ can be decomposed into the following four classes:

$$\begin{aligned}
\mathcal{C}_1 = \{i_1, i_{1+1}, K_0, i_n - 1, i_n\} &\sim \bullet\bullet K_0 \bullet\bullet \\
\mathcal{C}_2 = \{i_1, i_{1+1}, K_0, i_{n-1} < i_n - 1, i_n\} &\sim \bullet\bullet K_1 \circ\bullet \\
\mathcal{C}_3 = \{i_1, i_2 > i_1 + 1, K_0, i_n - 1, i_n\} &\sim \bullet\circ K_1' \bullet\bullet \\
\mathcal{C}_4 = \{i_1, i_2 > i_1 + 1, K_0, i_{n-1} < i_n - 1, i_n\} &\sim \bullet\circ K_2 \circ\bullet
\end{aligned} \tag{4.45}$$

Here K_p (with or without prime) stands for an element of $\mathcal{C}^{(n-5+p,\ k+1-p)}$. On the rhs we have introduced a symbolic description of the cluster in which the first two entries refer to the sites i_1 and $i_1 + 1$ ($\bullet$ means that the corresponding site is included in the cluster and $\circ$ means that it is not) and the last two give the information relative to the sites $i_n - 1$ and i_n. Everything in between is specified by the K_i's. Note that the clusters of the type $\mathcal{C}_4$

do not contribute to $B^{(1)}_{n-1,k+1}$ due to the constraint encoded in the delta function. With the above schematic description for the clusters, we have:

$$B^{(1)}_{n-1,k+1} = -4i \sum_{K_p \in \mathcal{C}^{(n-5+p,k+1-p)}} [f_{n-1}(\bullet \circ K_0 \bullet \bullet) - f_{n-1}(\bullet \bullet K_0 \circ \bullet) + f_{n-1}(\bullet \circ K_1 \circ \bullet) - f_{n-1}(\bullet \circ K_1' \circ \bullet)]. \tag{4.46}$$

Since the summation of terms containing K_1 and K_1' runs over identical sets, their contributions cancel out, leaving us with

$$B^{(1)}_{n-1,k+1} = -4i \sum_{K_0 \in \mathcal{C}^{(n-5,k+1)}} [f_{n-1}(\bullet \circ K_0 \bullet \bullet) - f_{n-1}(\bullet \bullet K_0 \circ \bullet)]. \tag{4.47}$$

We now turn to $B^{(2)}_{n-1,k+1}$. The crucial step in the present argument is the rewriting of (4.44) in the form

$$B^{(2)}_{n-1,k+1} = 2i[Pr \sum_{\mathcal{C} \in \mathcal{C}^{(n-1,k+1)}} \sum_{j=2}^{n-2} f_{n-1}(\mathcal{C}_{\widehat{i_j}}) - Pr \sum_{\mathcal{C} \in \mathcal{C}^{(n-1,k+1)}} \sum_{j=2}^{n-2} f_{n-1}(\mathcal{C}_{\widehat{i_{j+1}}})]. \tag{4.48}$$

Notice first that the constraint $i_{j+1} = i_j + 1$ has been relaxed. The symbol Pr indicates a projection operation whose action amounts to rescale the multiplicity of a given cluster to one. The rationale for this projection is as follows. In (4.44), the clusters $\mathcal{C}_{\widehat{i_j}}$ for a fixed j are all distinct due to the constraint $i_{j+1} = i_j + 1$. But when the constraint is relaxed, this is no longer so and it is then necessary to rule out repetitions. Here is an example. Translational invariance being not an issue here, consider a subset of $\mathcal{C}^{(n,k)}$, consisting of clusters with fixed extremities (and this is what the tilde denotes below). Let $n = 4, k = 1, i_1 = 1, i_4 = 5$:

$$\begin{aligned} \{\tilde{\mathcal{C}}\} &= \{\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}\}, \\ \{\tilde{\mathcal{C}}_{\widehat{i_2}}\}_{i_3 = i_2+1} &= \{\{1,3,5\},\{1,4,5\}\}, \\ \{\tilde{\mathcal{C}}_{\widehat{i_3}}\}_{i_3 = i_2+1} &= \{\{1,2,5\},\{1,3,5\}\}, \end{aligned} \tag{4.49}$$

so that

$$\sum_{\tilde{\mathcal{C}}} [f_3(\tilde{\mathcal{C}}_{\widehat{i_2}}) - f_3(\tilde{\mathcal{C}}_{\widehat{i_3}})]\delta_{i_3,i_2+1} = f_3(\{1,4,5\}) - f_3(\{1,2,5\}). \tag{4.50}$$

On the other hand, since

$$\begin{aligned} \{\tilde{\mathcal{C}}_{\widehat{i_2}}\} &= \{\{1,3,5\},\{1,4,5\},\{1,4,5\}\}, \\ \{\tilde{\mathcal{C}}_{\widehat{i_3}}\} &= \{\{1,2,5\},\{1,2,5\},\{1,3,5\}\}, \end{aligned} \tag{4.51}$$

we see that

$$\sum_{\tilde{\mathcal{C}}}[f_3(\tilde{\mathcal{C}}_{\widehat{i_2}}) - f_3(\tilde{\mathcal{C}}_{\widehat{i_3}})] = 2[f_3(\{1,4,5\}) - f_3(\{1,2,5\})]. \tag{4.52}$$

By working out another simple example, with $n = 5, k = 2$, the reader will convince himself that the projection has to be done for each sum separately before evaluating their difference.

We now prove the equivalence between (4.44) and (4.48). Clearly (4.48) incorporates all the terms in (4.44), which are those associated to clusters containing both i_j and $i_j + 1$. The extra terms in (4.48) turn out to cancel two by two: the contribution of a positive term, with $i_j \in \mathcal{C}$ and $i_j + 1 \notin \mathcal{C}$ is canceled by the contribution of a negative term associated to a cluster $\mathcal{C}'$ which differs from $\mathcal{C}$ only in that $i_j + 1 \in \mathcal{C}'$ and $i_j \notin \mathcal{C}'$. Schematically, if

$$\begin{aligned} \mathcal{C} &\sim K \bullet \circ K' \qquad (\bullet \quad \text{at site} \quad j), \\ \mathcal{C}' &\sim K \circ \bullet K' \qquad (\bullet \quad \text{at site} \quad j+1), \end{aligned} \tag{4.53}$$

then

$$\mathcal{C}_{\widehat{i_j}} = \mathcal{C}'_{\widehat{i_{j+1}}}. \tag{4.54}$$

Having established (4.48), we now rewrite it in the form

$$B^{(2)}_{n-1,k+1} = 2i[Pr \sum_{\mathcal{C}\in\mathcal{C}^{(n-1,k+1)}} f_{n-1}(\mathcal{C}_{\widehat{i_2}}) - Pr \sum_{\mathcal{C}\in\mathcal{C}^{(n-1,k+1)}} f_{n-1}(\mathcal{C}_{\widehat{i_{n-1}}})]. \tag{4.55}$$

The sum over clusters can again be separated into sums over the four classes (4.45), with the result:

$$\begin{aligned} B^{(2)}_{n-1,k+1} = 2i\ Pr \sum_{K_i\in\mathcal{C}^{(n-5+i,k+1-i)}} & [f_{n-1}(\bullet \circ K_0 \bullet \bullet) + f_{n-1}(\bullet \circ K_1 \circ \bullet) \\ & + f_{n-1}(\bullet \circ K'_{1\widehat{i_2}} \bullet \bullet) + f_{n-1}(\bullet \circ K_{2\widehat{i_2}} \circ \bullet)] \\ -2i\ Pr \sum_{K_i\in\mathcal{C}^{(n-5+i,k+1-i)}} & [f_{n-1}(\bullet \bullet K_0 \circ \bullet) + f_{n-1}(\bullet \bullet K_{1\widehat{i_{n-1}}} \circ \bullet) \\ & + f_{n-1}(\bullet \circ K'_1 \circ \bullet) + f_{n-1}(\bullet \circ K_{2\widehat{i_{n-1}}} \circ \bullet)]. \end{aligned} \tag{4.56}$$

The second and the seventh terms cancel and similarly for the fourth and the eight ones. The contribution of the third term being already included in that of the first one, it can be ignored (recall that due to the projection factor, multiplicities are irrelevant). Similarly, the sixth term can be dropped. With these simplifications, $B^{(2)}_{n-1,k+1}$ take the form

$$B^{(2)}_{n-1,k+1} = 2i \sum_{K_0\in\mathcal{C}^{(n-5,k+1)}} [f_{n-1}(\bullet \circ K_0 \bullet \bullet) - f_{n-1}(\bullet \bullet K_0 \circ \bullet)], \tag{4.57}$$

the projection operator being now unnecessary. Adding (4.47) and (4.57) yields

$$B_{n-1,k+1} = -2i \sum_{K_0 \in \mathcal{C}^{(n-5,k+1)}} [f_{n-1}(\bullet \circ K_0 \bullet \bullet) - f_{n-1}(\bullet \bullet K_0 \circ \bullet)]. \tag{4.58}$$

The trivial modification

$$\begin{aligned} B_{n-1,k+1} = -2i \sum_{K_0, K_{-1}} [&f_{n-1}(\bullet \circ K_0 \bullet \bullet) + f_{n-1}(\bullet \bullet K_{-1} \bullet \bullet) \\ &- f_{n-1}(\bullet \bullet K_0 \circ \bullet) - f_{n-1}(\bullet \bullet K_{-1} \bullet \bullet)], \end{aligned} \tag{4.59}$$

allows us to reach the form

$$\begin{aligned} B_{n-1,k+1} &= -2i \sum_{\mathcal{C} \in \mathcal{C}^{(n-1,k+1)}} [f_{n-1}(\mathcal{C}\ \bullet) - f_{n-1}(\bullet\ \mathcal{C})] \\ &= -2i \sum_{\mathcal{C} \in \mathcal{C}^{(n-1,k+1)}} [f_{n-1}(\mathcal{C}_{\widehat{i_n+1}}) - f_{n-1}(_{\widehat{i_1-1}}\mathcal{C})] \\ &= -A_{n-1,k+1}, \end{aligned} \tag{4.60}$$

which thereby proves the desired result.

h) Mutual commutativity of the H_n's and relation with the Q_n basis

Having established the commutativity of the H_n's with the hamiltonian, the final step in the proof of integrability is to show that all these conservation laws commute among themselves. For this, we will proceed indirectly by first expressing the charges H_n in the $\{Q_n\}$ basis defined recursively by

$$[B, Q_n] = Q_{n+1} \tag{4.61}$$

with $Q_2 = H_2$, and then use the fact that the Q_n mutually commute, being directly related to the commuting transfer matrices. We first evaluate the commutator of H_n ($n \geq 2$) with the boost operator. The result is:

$$[B, H_n] = \sum_{k=0}^{\max(1,[n/2]-1)} \beta_k^{(n)} H_{n+1-2k}, \tag{4.62}$$

where.

$$\begin{aligned} \beta_0^{(n)} &= n-1, \\ \beta_1^{(n>2)} &= 5-3n, \\ \beta_{1<\ell<[n/2]}^{(n)} &= -(n-2\ell-1)\alpha_{\ell,1} \end{aligned} \tag{4.63}$$

The simplest way of obtaining this formula is to compare the coefficients of the terms $F_{n-2k+1,0}$ on both sides. With this result, it is simple to obtain the expression for the charges Q_n in terms of the $\{H_n\}$'s. It is clear from (4.62) that for even (odd) n, Q_n can be expressed as a linear combination of the H_m with even (odd) $m \leq n$:

$$Q_n = \sum_{p=0}^{[n/2]-1} \gamma_p^{(n)} H_{n-2p}. \tag{4.64}$$

The coefficients γ can be determined from a simple recurrence relation:[13]

$$\gamma_\ell^{(n+1)} = \sum_{\substack{p,m\geq 0 \\ p+m=\ell}} \gamma_p^{(n)} \beta_m^{(n-2p)}, \tag{4.65}$$

with $\gamma_p^{(2)} = \delta_{p,0}$. For example, up to additive constants,

$$\begin{aligned} Q_4 &= 2H_4 - 4H_2, \\ Q_5 &= 6H_5 - 18H_3, \\ Q_6 &= 24H_6 - 96H_4 + 72H_2. \end{aligned} \tag{4.66}$$

Therefore, we see that $\{H_n\}$ represents a basis of the space of the conservation laws of the XXX chain, which can be obtained taking linear combinations of $\{Q_n\}$'s. Mutual commutativity of H_n's now follows trivially from that of the Q_n's.

i) An alternative finale

It is also possible to prove the involutive nature of H_n without using the transfer matrix formalism at all, by applying an inductive argument using (4.62) and the Jacobi identity. We sketch this argument below. First, we note that (4.62) can be rewritten in the form

$$H_{n+1} = \frac{1}{(n-1)}[B, H_n] + R_n, \tag{4.67}$$

where R_n is a linear combination of the charges $H_{m<n}$. Then, assuming that $[H_n, H_m] = 0$ for all $n, m < n_0$, we prove that $[H_{n_0+1}, H_k] = 0$, for $k < n_0$. For $k = 2$, the proof is the calculation given above in sections (b) to (g). For $k = 3$, the commutativity of H_{n_0+1} and H_3 can be established using the Jacobi identity and the fact that $[H_{n_0+2}, H_2] = 0$. Similarly, one may successively show that $[H_{n_0+1}, H_{k>3}] = 0$ using the Jacobi identity and the relations $[H_{n_0+\ell}, H_2] = 0$ for $\ell = 1, \ldots, k-1$. The above method constitutes an alternative purely algebraic proof of the integrability of the XXX model.

[13] Notice that the formulae for γ reflect our choice of normalization in (2.21); they need to be modified in another normalization, to take into account a rescaling of the Q_n's, corresponding to a rescaling of the spectral parameter.

5. The XYZ model revisited

5.1. Proof of the general pattern for the charges

We now present the proof that the charges of the XYZ model have the form (3.19). We will again proceed directly, calculating $[B, \tilde{f}_n(\mathcal{S})]$, for an arbitrary sequence $\mathcal{S} = \mathcal{C}^\pi$ with cluster $\mathcal{C} = \{i_1, \ldots, i_n\}$ and pattern $\pi = \{a_1, \ldots, a_n\}$. This commutator contains only terms of the form $\tilde{f}_{n\pm1}(\mathcal{S}')$, where $\mathcal{S}'$ can be obtained from $\mathcal{S}$ by a few simple transformations, given below:

$$\begin{aligned}
\sigma^{a_0}_{i_1-1}\mathcal{S} &\equiv \{i_1-1, i_1, \ldots, i_n\}^{\{a_0, a_0\times a_1, \ldots, a_n\}},\\
\mathcal{S}\sigma^{a_{n+1}}_{i_n+1} &\equiv \{i_1, \ldots, i_n, i_n+1\}^{\{a_1, \ldots, a_{n+1}\times a_n, a_{n+1}\}},\\
\mathcal{S}_{\widehat{\sigma^{a_k}_{i_k}}} &\equiv \{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n\}^{\{a_1, \ldots, a_{k-2}, a_k\times a_{k-1}, a_{k+1}, \ldots, a_n\}},\\
\widehat{\sigma^{a_k}_{i_k}}\mathcal{S} &\equiv \{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n\}^{\{a_1, \ldots, a_{k-1}, a_k\times a_{k+1}, a_{k+2}, \ldots, a_n\}},\\
\mathcal{S}_{\sigma^{a_j}_{i_j}\to\sigma^{a_k}_{i_k}\sigma^{a_\ell}_{i_\ell}} &\equiv \{i_1, \ldots, i_{j-1}, (i_k, i_\ell), i_{j+1}, \ldots, i_n\}^{\{a_1, \ldots, a_{j-1}, (a_k, a_\ell), a_{j+1}, \ldots, a_n\}}.
\end{aligned} \tag{5.1}$$

The following conventions will be adopted:

$$\begin{aligned}
\{a_1, \ldots, 0, \ldots, a_n\} &= 0,\\
\{a_1, \ldots, -a_k, \ldots, a_n\} &= -\{a_1, \ldots, a_k, \ldots, a_n\},
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
\tilde{f}_n(C^0) &= 0,\\
\tilde{f}_n(\mathcal{C}^{-\pi}) &= -\,\tilde{f}_n(\mathcal{C}^\pi).
\end{aligned} \tag{5.3}$$

The basic commutators of the form $[j\sigma_j\sigma_{j+1}, \tilde{f}_n(\mathcal{S})]$ may be evaluated similarly as in the XXX case. Summing up all these terms yields:

$$[B, \tilde{f}_n(\mathcal{S})] = b^+(\mathcal{S}) + b^{-+}(\mathcal{S}) + b^{+-}(\mathcal{S}) + b^-(\mathcal{S}), \tag{5.4}$$

where

$$b^\alpha(\mathcal{S}) = \sum_{e=x,y,z} b^\alpha_e(\mathcal{S}), \qquad \alpha \in \{+, -+, +-, -\}, \tag{5.5}$$

and

$$
\begin{aligned}
b_e^+(\mathcal{S}) =& -(i_1-1)\tilde{f}_{n+1}(\sigma^e_{i_1-1}\mathcal{S}) + i_n\tilde{f}_{n+1}(\mathcal{S}\sigma^e_{i_n+1}),\\
b_e^{-+}(\mathcal{S}) =& -2i_1\tilde{f}_{n-1}(\mathcal{S}_{\widehat{\sigma^e_{i_2}}})\delta_{i_1+1,i_2} + 2i_{n-1}\tilde{f}_{n-1}(_{\widehat{\sigma^e_{i_{n-1}}}}\mathcal{S})\delta_{i_{n-1}+1,i_n}\\
&+\sum_{j=2}^{n-2} i_j[\tilde{f}_{n-1}(_{\widehat{\sigma^e_{i_j}}}\mathcal{S}) - \tilde{f}_{n-1}(\mathcal{S}_{\widehat{\sigma^e_{i_{j+1}}}})]\delta_{i_{j+1},i_j+1},\\
b_e^{+-}(\mathcal{S}) =& i_1(\lambda_e)^2\tilde{f}_{n+1}(\mathcal{S}_{\sigma^{a_1}_{i_1}\to\sigma^{e\times a_1}_{i_1+1}\sigma^e_{i_1}})(1-\delta_{i_1,i_2-1})\\
&-(i_n-1)(\lambda_e)^2\tilde{f}_{n+1}(\mathcal{S}_{\sigma^{a_n}_{i_n}\to\sigma^e_{i_n-1}\sigma^{e\times a_n}_{i_n}})(1-\delta_{i_n,i_{n-1}+1})\\
&+\sum_{j=2}^{n-2} i_j(\lambda_e)^2\tilde{f}_{n-1}(\mathcal{S}_{\sigma^{a_j}_{i_j}\to\sigma^{e\times a_j}_{i_j}\sigma^e_{i_j+1}})(1-\delta_{i_{j+1},i_j+1})\\
&+\sum_{j=3}^{n-1}(i_j-1)(\lambda_e)^2\tilde{f}_{n+1}(\mathcal{S}_{\sigma^{a_j}_{i_j}\to\sigma^e_{i_j-1}\sigma^{e\times a_j}_{i_j}})(1-\delta_{i_{j-1},i_j-1}),\\
b_e^-(\mathcal{S}) =& 2i_1(\lambda_e)^2\tilde{f}_{n-1}(_{\widehat{\sigma^e_{i_1}}}\mathcal{S})\delta_{i_1+1,i_2} - 2i_{n-1}(\lambda_e)^2\tilde{f}_{n-1}(\mathcal{S}_{\widehat{\sigma^e_{i_n}}})\delta_{i_{n-1}+1,i_n}.
\end{aligned}
\tag{5.6}
$$

With the exception of the third and fourth terms in $b^{+-}(\mathcal{S})$, all the terms in (5.4) contain only expressions corresponding to ordered sequences. Fortunately, thanks to the factors arising from the coefficients $g(\mathcal{S})$, the contributions due to disordered clusters will cancel[14] in summation over all possible clusters for a given pattern π. Therefore, the commutator $[B, \tilde{F}^n_{m,k}]$ can be expressed as a linear combination of the terms of the type $\tilde{f}(\mathcal{S}')$, with $\mathcal{S}'$ being an ordered sequence.

A simple inductive step finishes the proof. Assume that the n-th conserved charge is of the form (3.19). As we have just shown, the $n+1$-th charge Q_{n+1}, given by $[B, Q_n]$, is a linear combination of the polynomials $\tilde{f}(\mathcal{S})$. Now, because the commutator $[B, Q_n]$ is translation invariant, it must in fact be a linear combination of elements of the spaces $\mathcal{F}_{n+1-2k,l}$. Therefore, the charge Q_{n+1} is again necessarily of the form (3.19).

5.2. Recursion relations for the $\tilde{F}$'s

Using the operations (5.6) (which naturally extend to any space $\mathcal{F}_{n,m}$), one can derive recursion relations for the $\tilde{F}$'s. It is convenient to represent this recursion as a transformation of the triangle corresponding to Q_n into a new triangle corresponding to Q_{n+1} (cf. Fig. 2.).

[14] This can be seen by an argument similar to the one used in section (4.2.d) to demonstrate the cancellation of the r-terms in the XXX case.

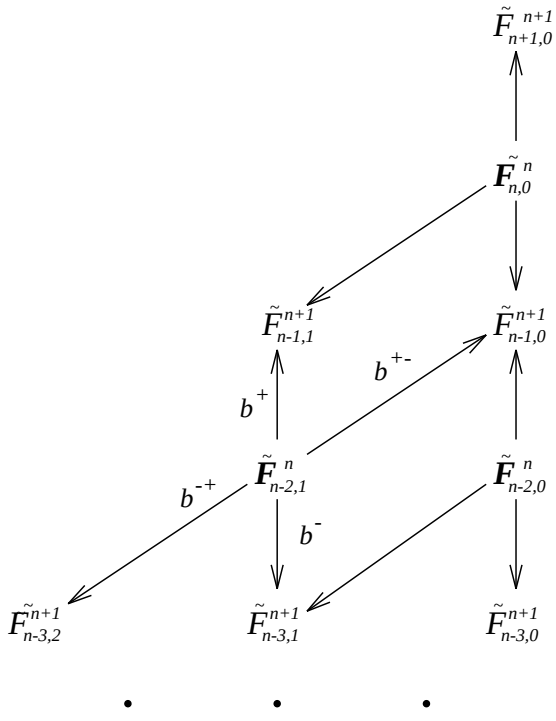

Fig. 2. The action of the boost operator on the triangle representing Q_n for the XYZ model.

It is clear that the boost operator does not modify the structure of the triangle if n is even (i.e. the triangles corresponding to Q_{2m} and Q_{2m+1} differ only in labeling of points). For n odd, the action of the boost produces a new strip at the bottom of the triangle (therefore adding $[(n+1)/2]$ new points). The left edge of the triangle representing Q_{n+1} is given by:

$$\tilde{F}^{n+1}_{n+1-2k,0<k<[(n+1)/2]} = b^{-+}(\tilde{F}^{n}_{n-2(k-1),k-1}) + b^{+}(\tilde{F}^{n}_{n-2k,k}), \tag{5.7}$$

with

$$\tilde{F}^{n+1}_{n+1,0} = b^{+}(\tilde{F}^{n}_{n,0}). \tag{5.8}$$

The right edge of the Q_{n+1} triangle is:

$$\tilde{F}^{n+1}_{n+1-2k,0} = b^{-}(\tilde{F}^{n}_{n-2(k-1),0}) + b^{+}(\tilde{F}^{n}_{n-2k,0}) + b^{+-}(\tilde{F}^{n}_{n-2k,1}). \tag{5.9}$$

The interior of the triangle can be obtained from:

$$\begin{aligned}\tilde{F}^{n+1}_{n+1-2k,0<\ell<k} =& b^{-}(\tilde{F}^{n}_{n-2(k-1),\ell}) + b^{-+}(\tilde{F}^{n}_{n-2(k-1),\ell-1}) \\ &+ b^{+}(\tilde{F}^{n}_{n-2k,\ell}) + b^{+-}(\tilde{F}^{n}_{n-2k,\ell+1}).\end{aligned} \tag{5.10}$$

Finally, for n odd, the bottom edge of the triangle Q_{n+1} is:

$$\tilde{F}^{n+1}_{2,0<\ell<[(n+1)/2]} = b^{-}(\tilde{F}^{n}_{3,\ell}) + b^{-+}(\tilde{F}^{n}_{3,\ell-1}). \tag{5.11}$$

These relations provide a recursive way to calculate the conserved charges. As in the isotropic case, the basis $\{Q_n\}$ contains terms proportional to lower order charges; it is more convenient to express the results in another basis, denoted $\{H_n\}$, obtained by taking appropriate linear combinations of the form (3.22), in which lower order contributions are subtracted.

5.3. Explicit form of the first few conserved charges

We present below several of the lowest-order conservation laws of the XYZ model. The first non-trivial charge beyond the hamiltonian is:

$$H_3 = \tilde{F}^{3}_{3,0} = \sum_{j\in\Lambda} (\hat{\boldsymbol{\sigma}}_j \times \tilde{\boldsymbol{\sigma}}_{j+1}) \cdot \hat{\boldsymbol{\sigma}}_{j+2}. \tag{5.12}$$

The four-spin charge is:

$$H_4 = \tilde{F}^{4}_{4,0} + \tilde{F}^{4}_{2,1} + \tilde{F}^{4}_{2,0}, \tag{5.13}$$

where

$$\begin{aligned}\tilde{F}^{4}_{4,0} =& \sum_{j\in\Lambda} ((\hat{\boldsymbol{\sigma}}_j \times \tilde{\boldsymbol{\sigma}}_{j+1}) \times \tilde{\boldsymbol{\sigma}}_{j+2}) \cdot \hat{\boldsymbol{\sigma}}_{j+3}, \\ \tilde{F}^{4}_{2,1} =& \sum_{j\in\Lambda} \sum_{a\in\{x,y,z\}} (\frac{\lambda_x\lambda_y\lambda_z}{\lambda_a}) \hat{\sigma}^a_j \hat{\sigma}^a_{j+2}, \\ \tilde{F}^{4}_{2,0} =& \sum_{j\in\Lambda} \sum_{a\in\{x,y,z\}} (\lambda_a)^2 \hat{\sigma}^a_j \hat{\sigma}^a_{j+1}.\end{aligned} \tag{5.14}$$

The five-spin charge corresponds to a triangle with the same structure as that of H_4, namely:

$$H_5 = \tilde{F}^{5}_{5,0} + \tilde{F}^{5}_{3,1} + \tilde{F}^{5}_{3,0}, \tag{5.15}$$

with

$$\begin{aligned}
\tilde{F}^5_{5,0} =& \sum_{j\in\Lambda}(((\hat{\boldsymbol{\sigma}}_j \times \tilde{\boldsymbol{\sigma}}_{j+1}) \times \tilde{\boldsymbol{\sigma}}_{j+2}) \times \tilde{\boldsymbol{\sigma}}_{j+3}) \cdot \hat{\boldsymbol{\sigma}}_{j+4},\\
\tilde{F}^5_{3,1} =& \sum_{j\in\Lambda}\sum_{a\in\{x,y,z\}}(\frac{\lambda_x\lambda_y\lambda_z}{\lambda_a})\epsilon^{abc}(\hat{\sigma}^a_j\tilde{\sigma}^b_{j+2}\hat{\sigma}^c_{j+3} + \hat{\sigma}^b_j\tilde{\sigma}^c_{j+1}\hat{\sigma}^a_{j+3}),\\
\tilde{F}^5_{3,0} =& -\sum_{j\in\Lambda}\sum_{a\neq b\neq c}(\lambda_b)^2\epsilon^{abc}\hat{\sigma}^a_j\tilde{\sigma}^b_{j+1}\hat{\sigma}^c_{j+2}.
\end{aligned} \tag{5.16}$$

The six-spin "monster" is:

$$H_6 = \tilde{F}^6_{6,0} + \tilde{F}^6_{4,1} + \tilde{F}^6_{2,2} + \tilde{F}^6_{4,0} + \tilde{F}^6_{2,1} + \tilde{F}^6_{2,0}, \tag{5.17}$$

with

$$\begin{aligned}
\tilde{F}^6_{6,0} =& \sum_{j\in\Lambda}(((\hat{\boldsymbol{\sigma}}_j \times \tilde{\boldsymbol{\sigma}}_{j+1}) \times \tilde{\boldsymbol{\sigma}}_{j+2}) \times \tilde{\boldsymbol{\sigma}}_{j+3}) \times \tilde{\boldsymbol{\sigma}}_{j+4}) \cdot \tilde{\boldsymbol{\sigma}}_{j+5},\\
\tilde{F}^6_{4,1} =& \sum_{j\in\Lambda}\sum_{a\neq b}[(\frac{\lambda_x\lambda_y\lambda_z}{\lambda_a})(\hat{\sigma}^a_j\tilde{\sigma}^b_{j+2}\tilde{\sigma}^a_{j+3}\hat{\sigma}^b_{j+4} - \hat{\sigma}^a_j\tilde{\sigma}^b_{j+2}\tilde{\sigma}^b_{j+3}\hat{\sigma}^a_{j+4}\\
&+ \hat{\sigma}^b_j\tilde{\sigma}^a_{j+1}\tilde{\sigma}^b_{j+2}\hat{\sigma}^a_{j+4} - \hat{\sigma}^a_j\tilde{\sigma}^b_{j+1}\tilde{\sigma}^b_{j+2}\hat{\sigma}^a_{j+4})\\
&+ \lambda_a\lambda_b(\hat{\sigma}^a_j\tilde{\sigma}^b_{j+1}\tilde{\sigma}^a_{j+3}\hat{\sigma}^b_{j+4} - \hat{\sigma}^a_j\tilde{\sigma}^b_{j+1}\tilde{\sigma}^b_{j+3}\hat{\sigma}^a_{j+4})],\\
\tilde{F}^6_{2,2} =& \sum_{j\in\Lambda}\sum_{a\in\{x,y,z\}}(\frac{\lambda_x\lambda_y\lambda_z}{\lambda_a})^2\hat{\sigma}^a_j\hat{\sigma}^a_{j+3},\\
\tilde{F}^6_{4,0} =& \sum_{j\in\Lambda}\sum_{a\neq b}(\lambda_b^2 - \lambda_a^2)\hat{\sigma}^a_j\tilde{\sigma}^b_{j+1}\tilde{\sigma}^b_{j+3}\hat{\sigma}^a_{j+4},\\
\tilde{F}^6_{2,1} =& \sum_{j\in\Lambda}\sum_{a\in\{x,y,z\}}\lambda_a^2(\frac{\lambda_x\lambda_y\lambda_z}{\lambda_a})\hat{\sigma}^a_j\hat{\sigma}^a_{j+2},\\
\tilde{F}^6_{2,0} =& \sum_{j\in\Lambda}\sum_{a\neq b\neq c}(\lambda_a^2 + \lambda_b^2)(\lambda_a^2 + \lambda_c^2)\hat{\sigma}^a_j\hat{\sigma}^a_{j+1}.
\end{aligned} \tag{5.18}$$

The logarithmic derivatives of the transfer matrix can be expressed as linear combinations of the H_n's:

$$\begin{aligned}
Q_3 =& H_3,\\
Q_4 =& 2H_4 - 2p^2H_2,\\
Q_5 =& 6H_5 - 4p^2H_3,\\
Q_6 =& 24H_6 - 32p^2H_4 + 8p^4H_2.
\end{aligned} \tag{5.19}$$

with $p^2 = {\lambda_x}^2 + {\lambda_y}^2 + {\lambda_z}^2$. In the isotropic limit ($\lambda_x = \lambda_y = \lambda_z = 1$), the XYZ charges reduce to linear combinations of the XXX charges given by (4.2), which we denote below

as H_n^{XXX} to avoid misunderstanding:

$$\begin{aligned} H_3 =& H_3^{XXX}, \\ H_4 =& H_4^{XXX} + H_2^{XXX}, \\ H_5 =& H_5^{XXX} - H_3^{XXX}, \\ H_6 =& H_6^{XXX} + 4H_2^{XXX}. \end{aligned} \tag{5.20}$$

From (5.19) we then recover the relations (4.66).

6. The conserved charges in the XY model

6.1. Generalities

For completeness, we present in this section the results for the charges for the XY model:

$$H_{XY} = \sum_{j\in\Lambda} \lambda_x \sigma_j^x \sigma_{j+1}^x + \lambda_y \sigma_j^y \sigma_{j+1}^y. \tag{6.1}$$

As is well known, the XY spin chain is equivalent to a free fermion theory; the charges for this system have thus a particularly simple form, and they have been discussed by many authors [26-29]. Although the following results are not new, the derivation given here is different from the approach of [26-29]. An interesting feature of the XY case is that there exist two independent families of conservation laws. Both families persist when the model is perturbed by a perpendicular (i.e. in the z-direction) magnetic field.

Since the XY model is just a special case of the XYZ hamiltonian, we can apply to it the apparatus developed in sections (3) and (5). The general formula (3.19) greatly simplifies for the XY case. First, note that since $\lambda_z = 0$, the coefficient $g(\mathcal{S})$ vanishes for any sequence containing holes. Furthermore, $\tilde{f}_n(\mathcal{S}) = 0$ for sequences with $a_1 = z$ or $a_n = z$ or $a_2, \ldots, a_{n-1} = x, y$. Thus, $\tilde{f}_n(\mathcal{S})$ can be non-vanishing only for the sequences of the form:

$$\mathcal{S}_{n,j}^{\alpha\beta} = \{\sigma_j^\alpha, \sigma_{j+1}^z, \ldots, \sigma_{j+n-2}^z, \sigma_{j+n-1}^\beta\}, \tag{6.2}$$

with $\alpha, \beta \in \{x, y\}$. Observe further that

$$\tilde{f}_n(\mathcal{S}_{n,j}^{\alpha\beta}) = \epsilon\ e_n^{\alpha\beta}, \tag{6.3}$$

where

$$e_{n,j}^{\alpha\beta} = \sigma_j^\alpha \sigma_{j+1}^z \ldots \sigma_{j+n-2}^z \sigma_{j+n-1}^\beta, \tag{6.4}$$

and the coefficient ϵ assumes values ± 1 or 0. It vanishes when $\alpha \neq \beta$ if n is even, or $\alpha = \beta$, if n is odd. We also introduce the notation

$$e_n^{\alpha\beta} = \sum_{j\in\Lambda} e_{n,j}^{\alpha\beta} \tag{6.5}$$

The charges (3.19) for the XY model take thus the simple form:

$$Q_n = \sum_{k=0}^{[n/2]-1} \tilde{F}_{n-2k,0}^n, \tag{6.6}$$

with $\tilde{F}_{n-2k,0}^n \in \mathcal{F}_{n-2k,0}$. This shows that the triangle corresponding to Q_n collapses to its right edge. Note that the spaces $\mathcal{F}_{n-2k,0}$ are two-dimensional. A convenient basis is given by $\{e_{n-2k}^{xx}, e_{n-2k}^{yy}\}$ for n even, and $\{e_{n-2k}^{xy}, e_{n-2k}^{yx}\}$ for n odd. As we will shortly see, the family $\{Q_n\}$ contains only half of the integrals of motion of the XY model.

6.2. The XX case

Let us consider first the special case of the XX model with $\lambda_x = \lambda_y$. One finds two conserved quantities for each n, which define two families:

$$\begin{aligned} H_n^{(+)} &= e_n^{xx} + e_n^{yy} \qquad n \text{ even}, \\ &= e_n^{xy} - e_n^{yx} \qquad n \text{ odd}, \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} H_n^{(-)} &= e_n^{xy} - e_n^{yx} \qquad n \text{ even}, \\ &= e_n^{xx} + e_n^{yy} \qquad n \text{ odd}. \end{aligned} \tag{6.8}$$

In particular, there are two two-spin hamiltonians;

$$H_2^{(+)} = \sum_{j\in\Lambda} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y, \tag{6.9}$$

$$H_2^{(-)} = \sum_{j\in\Lambda} \sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x. \tag{6.10}$$

$H_2^{(+)}$ is just the XX hamiltonian, while $H_2^{(-)}$ is a special case of the Dzyaloshinski-Moriya interaction [35].

Commutativity of $H_n^{(\pm)}$ with both hamiltonians $H_2^{(\pm)}$ can be easily verified directly by a short calculation. Furthermore, the boost operator,

$$B = \frac{1}{2i} \sum_{j\in\Lambda} j[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y], \tag{6.11}$$

has ladder properties:

$$[B, H_n^{(+)}] = (n-1)(-1)^{n-1}(H_{n+1}^{(+)} + H_{n-1}^{(+)}), \tag{6.12}$$

(valid for all n with the understanding that $H_1^{(+)} = 0$). Since the XX hamiltonian $H_2^{(+)}$ is proportional to to the first logarithmic derivative of the transfer matrix Q_2, it follows that $H_n^{(+)}$ can be expressed as a linear combination of the logarithmic derivatives of T. This shows the mutual commutativity of the $H_n^{(+)}$'s. Note that the family $\{H_n^{(+)}\}$ can be obtained from the family $\{Q_n\}$ by a transformation of the form (3.22). Clearly, the triangle corresponding to $H_n^{(+)}$ degenerates to a single point. In terms of the associated triangles, this transformation corresponds then to redefining to zero all the points on the right edge, except for the vertex $F_{n,0}^n$.

It is clear from (6.12) that for even (odd) n, Q_n can be expressed as a linear combination of the $H_m^{(+)}$ with even (odd) $m \leq n$:

$$Q_n = \sum_{p=0}^{[n/2]-1} \gamma_p^{(n)} H_{n-2p}^{(+)}. \tag{6.13}$$

The coefficients γ satisfy the recurrence relation (4.65), where the coefficients β are now: $\beta_0^{(n)} = \beta_1^{(n)} = (-1)^{(n-1)}(n-1)$. For example, up to additive constants,

$$\begin{aligned} Q_3 &= -H_3^{(+)}, \\ Q_4 &= 2H_4^{(+)} - 2H_2^{(+)}, \\ Q_5 &= 6H_5^{(+)} + 8H_3^{(+)}, \\ Q_6 &= 24H_6^{(+)} + 40H_4^{(+)} + 16H_2^{(+)}. \end{aligned} \tag{6.14}$$

(6.14) can be also obtained as a special case from the formulae (5.12)-(5.17) for the general XYZ model.

In contrast, the charges of the second family $H_n^{(-)}$ are not of the general form (3.19). Remarkably, the boost operator associated to $H_2^{(+)}$ acts as a ladder operator for the second family too (as first noticed in [36]):

$$[B, H_n^{(-)}] = (n-1)(-1)^{n}(H_{n+1}^{(-)} + H_{n-1}^{(-)}), \tag{6.15}$$

with $H_1^{(-)} = -2\sum_{j\in\Lambda}\sigma_j^z$. Using (6.15), (6.12), together with the Jacobi identity one can prove by an inductive argument similar to the one in section (4.2), that the charges of the second family commute among themselves and with those of the first family.

Note that under a spin rotation by $\pi/2$ around the z-axis but only on even sites, i.e.:

$$\begin{aligned} \sigma^x_{2j} &\to \sigma^y_{2j}, \qquad \sigma^y_{2j} \to -\sigma^x_{2j} \\ \sigma^x_{2j+1} &\to \sigma^x_{2j+1}, \qquad \sigma^y_{2j+1} \to \sigma^y_{2j+1} \end{aligned} \tag{6.16}$$

the two-spin charge $H_2^{(-)}$ transforms into the staggered XX hamiltonian:

$$H_2^{(-)} \to \sum_{j\in\Lambda} (-1)^j (\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1}). \tag{6.17}$$

Both $H_n^{(+)}$ and $H_n^{(-)}$ behave as scalars under a global spin rotation around the z-axis. Indeed, it is simple to check that all members of both families commute with the generator of such rotation, that is the z component of the total spin,

$$S^z = \sum_{j\in\Lambda} \sigma^z_j. \tag{6.18}$$

Therefore, both families survive when the model is perturbed by a magnetic field term hS^z. On the other hand, while $H_n^{(+)}$ is invariant under global parity transformations, i.e. $\sigma^a_j \to -\sigma^a_j$, $H_n^{(-)}$ changes sign and for this reason we qualify it as a pseudoscalar.

6.3. The general XYh case

We turn now to the analysis of the general XY model incorporating the magnetic field term. The defining hamiltonian is thus

$$H_{XY} + hS^z = \sum_{j\in\Lambda} \lambda_x \sigma^x_j \sigma^x_{j+1} + \lambda_y \sigma^y_j \sigma^y_{j+1} + h\sigma^z_j. \tag{6.19}$$

As in the XX case, we look for the charges in terms of the basis $e_n^{\alpha\beta}$. We denote by $\mathcal{A}_n$ the span of $\{e_n^{xx}, e_n^{yy}\}$ for n even, and the span of $\{e_n^{xy}, e_n^{yx}\}$ for n odd (i.e. $\mathcal{A}_n = \mathcal{F}_{n,0}$). Similarly, $\mathcal{B}_n$ will denote the span of $\{e_n^{xy}, e_n^{yx}\}$ for n even, and that of $\{e_n^{xx}, e_n^{yy}\}$ for n odd. In addition, $\mathcal{A}_1 = 0$, $\mathcal{B}_1 = \mathrm{span}(S^z)$. Let $\mathcal{A} = \bigoplus_{n=1}^N \mathcal{A}_n$ and $\mathcal{B} = \bigoplus_{n=1}^N \mathcal{B}_n$. We now introduce two derivations on $\mathcal{A}\oplus\mathcal{B}$ given by:

$$\delta_{XY}(P) = \frac{1}{2i}[H_{XY}, P], \tag{6.20}$$

$$\delta_Z(P) = \frac{h}{2i}[S^z, P], \tag{6.21}$$

for $P \in \mathcal{A}\oplus\mathcal{B}$. The action of these operators can be visualized on the diagram in Fig. 3.

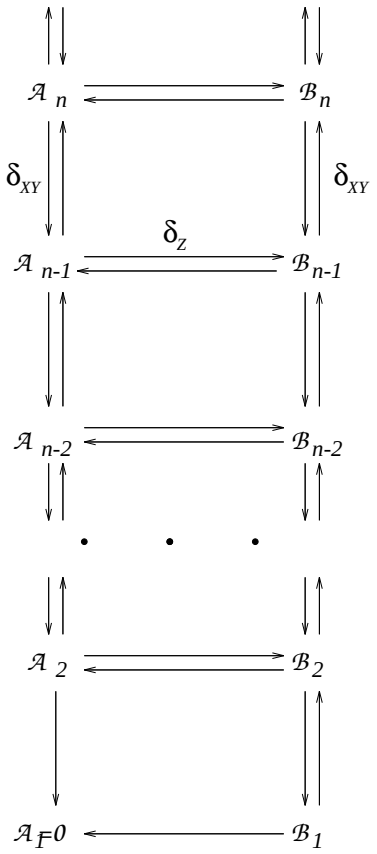

Fig. 3. The action of the derivations δ_{XY}, δ_Z in the space $\mathcal{A} \oplus \mathcal{B}$ of the XY model.

The derivation δ_Z acts "horizontally," i.e. $\delta_Z : \mathcal{A}_n \to \mathcal{B}_n$, $\delta_Z : \mathcal{B}_n \to \mathcal{A}_n$, while δ_{XY} acts "vertically," i.e. $\delta_{XY} : \mathcal{A}_n \to \mathcal{A}_{n-1} \oplus \mathcal{A}_{n+1}$, and $\delta_{XY} : \mathcal{B}_n \to \mathcal{B}_{n-1} \oplus \mathcal{B}_{n+1}$. The action of δ_{XY} on an arbitrary $X \in \mathcal{A} \oplus \mathcal{B}$ can be written as:

$$\delta_{XY}(X) = u(X) + d(X), \tag{6.22}$$

where $u : \mathcal{A}_n \to \mathcal{A}_{n+1}$, $u : \mathcal{B}_n \to \mathcal{B}_{n+1}$, and $d : \mathcal{A}_n \to \mathcal{A}_{n-1}$, $d : \mathcal{B}_n \to \mathcal{B}_{n-1}$. A simple calculation yields:

$$\begin{aligned} u(e_n^{xx}) =& -\lambda_y(e_{n+1}^{yx} + e_{n+1}^{xy}), \\ u(e_n^{yy}) =& \lambda_x(e_{n+1}^{yx} + e_{n+1}^{xy}), \\ u(e_n^{xy}) =& u(e_n^{yx}) = \lambda_x e_{n+1}^{xx} - \lambda_y e_{n+1}^{yy}, \end{aligned} \tag{6.23}$$

and

$$\begin{aligned} d(e_n^{xx}) =& -\lambda_x(e_{n-1}^{yx}+e_{n-1}^{xy}),\\ d(e_n^{yy}) =& \lambda_y(e_{n-1}^{yx}+e_{n-1}^{xy}),\\ d(e_n^{xy}) =& d(e_n^{yx}) = \lambda_y e_{n-1}^{xx} - \lambda_x e_{n-1}^{yy}. \end{aligned} \tag{6.24}$$

Similarly,

$$\begin{aligned} \delta_Z(e_n^{xx}) =& h(e_n^{yx}+e_n^{xy}),\\ \delta_Z(e_n^{yy}) =& -h(e_n^{yx}+e_n^{xy}),\\ \delta_Z(e_n^{xy}) =& \delta_Z(e_n^{yx}) = h(e_n^{yy}-e_n^{xx}). \end{aligned} \tag{6.25}$$

The kernel of u in the space $\mathcal{A}_n \oplus \mathcal{B}_n$ is thus two dimensional (it is the span of $e_n^{xy}-e_n^{yx}$ and $\lambda_x e_n^{xx}+\lambda_y e_n^{yy}$), allowing for the existence of two families of integrals of motion. The explicit expressions for half of the XX charges are not affected by the anisotropic deformation of the hamiltonian: it is simple to check, using (6.23), that

$$H_n^{(1)} = e_n^{xy} - e_n^{yx} \tag{6.26}$$

commutes with the hamiltonian, for any n. In order to find explicit expressions for the remaining charges, we write them in the form

$$H_n^{(2)} = \lambda_x e_n^{xx} + \lambda_y e_n^{yy} + \sum_{k=1}^{[n/2]} X_{n-2k} + \sum_{k=1}^{[(n+1)/2]} Y_{n+1-2k}, \tag{6.27}$$

with $X_{n-2k} \in \mathcal{A}_{n-2k}(\mathcal{B}_{n-2k})$ and $Y_{n+1-2k} \in \mathcal{B}_{n+1-2k}(\mathcal{A}_{n+1-2k})$ if n is even (odd). The condition $[H_{XY}+hS^z, H_n^{(2)}]=0$ gives then the following system of equations:

$$\begin{aligned} u(Y_{n-1}) + \delta_Z(\lambda_x e_n^{xx} + \lambda_y e_n^{yy}) &= 0,\\ u(X_{n-2k}) + \delta_Z(Y_{n-2k+1}) + d(X_{n-2k+2}) &= 0, \quad \text{for} \quad k=1,\dots,[(n+1)/2], \end{aligned} \tag{6.28}$$

which allow for a recursive determination of the X, Y's: using (6.23), Y_{n-1} is fixed by the first equation; the second one, with $k=1$, determines X_{n-2}; then, the case $k=2$ leads to Y_{n-3}, etc. Note that since the kernel of u in $\mathcal{A}_m \oplus \mathcal{B}_m$ is two-dimensional, (6.28) does not fix the terms X_{n-2k} and Y_{n+1-2k} uniquely, but only up to an admixture of lower order integrals of motion. In solving (6.28) one may thus impose additional constraints. One possibility is to demand that X_m and Y_m be orthogonal to the kernel of u in $\mathcal{A}_m \oplus \mathcal{B}_m$. This yields

$$H_n^{(2)} = \lambda_x e_n^{xx} + \lambda_y e_n^{yy} + \sum_{k=1}^{n-1} c_k(\lambda_y e_{n-k}^{xx} - \lambda_x e_{n-k}^{yy}), \tag{6.29}$$

where the coefficients c_k satisfy

$$c_k = \frac{1}{({\lambda_x}^2 + {\lambda_y}^2)}(h(\lambda_x + \lambda_y)c_{k-1} - 2\lambda_x\lambda_y c_{k-2}), \tag{6.30}$$

with

$$c_1 = \frac{h(\lambda_x - \lambda_y)}{({\lambda_x}^2 + {\lambda_y}^2)} \ , \quad \text{and} \quad c_0 = \frac{({\lambda_y}^2 - {\lambda_x}^2)}{({\lambda_x}^2 + {\lambda_y}^2)} \tag{6.31}$$

Alternatively, one may obtain a simpler solution by demanding that the number of terms X_{n-2k} and Y_{n+1-2k} in (6.27) be minimal. This leads to following expressions, that have first been written in [26]:

$$H_n^{(2)} = \lambda_x e_n^{xx} + \lambda_y e_n^{yy} + Y_{n-1} + X_{n-2}, \tag{6.32}$$

where

$$\begin{aligned} Y_{n-1} &= -\, h(e_{n-1}^{xx} + e_{n-1}^{yy}), \\ X_{n-2} &= \lambda_x e_{n-2}^{yy} + \lambda_y e_{n-2}^{xx}. \end{aligned} \tag{6.33}$$

As in the XX case, it can be shown that the boost acts as a ladder operator. The recursive application of the boost, starting from the hamiltonian (6.19), generates a family of scalar conserved charges. For $h = 0$ the n-th order charge of this family coincides with the n-th logarithmic derivative (2.18) of the XYZ transfer matrix, evaluated at $\lambda_z = 0$. The recursive application of the boost operator, starting from $H_2^{(-)}$ produces a family of pseudoscalar conservation laws.[15]

To end this section, let us mention that the degenerate case $\lambda_y = 0$ of the XYh model (which could be called the Xh model) is known as the Ising chain. It is evident that this system inherits from the XYh model two distinct families of conserved charges.

6.4. Higher order ladder operators

An interesting feature of the XYh model is the existence of higher order ladder operators [28, 36]. For simplicity we focus again on the XX case. The first moment of $H_n^{(+)}$ acts then as a ladder operator whose application to a charge of order m generates a charge of order $m + n - 1$. For example,

$$B_3^{(+)} = \frac{1}{2i}\sum_{j\in\Lambda} j[e_{3,j}^{xy} - e_{3,j}^{yx}], \tag{6.34}$$

[15] Note that for $h = 0$ the charges of second family cannot be expressed in terms of the logarithmic derivatives of the XYZ transfer matrix, in the parametrization adopted here. Observe however that both families can be obtained from the XYh transfer matrix, written down in [37].

acts in the following way:

$$\begin{aligned}[B_3^{(+)}, H_2^{(+)}] &= H_4^{(+)} - H_2^{(+)}, \\ [B_3^{(+)}, H_{n>2}^{(+)}] &= (n-1)(H_{n+2}^{(+)} - H_{n-2}^{(+)}).\end{aligned} \tag{6.35}$$

The first moments of the pseudoscalar charges transform scalar (pseudoscalar) charges into pseudoscalar (scalar) ones, e.g.

$$B_2^{(-)} = \frac{1}{2i}\sum_{j\in\Lambda} j[\sigma_j^x\sigma_{j+1}^y - \sigma_j^y\sigma_{j+1}^x], \tag{6.36}$$

acts as follows:

$$[B_2^{(-)}, H_n^{(\pm)}] = (n-1)(H_{n+1}^{(\mp)} - H_{n-1}^{(\mp)}). \tag{6.37}$$

Since the commutator of two ladder operators has again the ladder property, any two such operators can be used to generate an infinite family of ladder operators. No such higher order ladder operators are known in the general nondegenerate case of the XYZ model. In particular, the first moment of the density of the three-spin XYZ charge does not have the ladder property.

7. The $su(M)$ invariant spin chain

7.1. Formulation of the model

In this section we consider the isotropic $su(M)$ version of the XXX model in the fundamental representation. The hamiltonian of this system reads [38]:

$$H_2 = \sum_{j\in\Lambda} t_j^a t_{j+1}^a, \tag{7.1}$$

where t_j^a, $a = 1, ..., M^2-1$, are the $su(M)$ generators in the fundamental representation, acting non-trivially only on the $j-th$ factor of the Hilbert space $\bigotimes_j \mathcal{C}^N$. For $M = 2$ (7.1) is simply the hamiltonian of the spin-1/2 XXX model. It is convenient to choose the normalization so that t^a are the $su(M)$ Gell-Mann matrices:

$$\begin{aligned}[t^a, t^b] &= 2if^{abc}t^c, \\ t^a t^b + t^b t^a &= 4\delta_{ab}/M + 2d^{abc}t^c,\end{aligned} \tag{7.2}$$

where f^{abc} are the structure constants of $su(M)$, and d^{abc} is a completely symmetric tensor, non-trivial for all $M > 2$.

7.2. Expressions for the conserved charges

Remarkably, the conserved charges for general M have the same structure as in the $M = 2$ case (the ordinary $s = 1/2$ XXX model). The expressions for the general case can be obtained from (4.2) by a simple substitution:

$$\sigma^a \to t^a \qquad \epsilon^{abc} \to f^{abc}. \tag{7.3}$$

For example,

$$H_3 = \sum_{j\in\Lambda} f^{abc} t_j^a t_{j+1}^b t_{j+2}^c, \tag{7.4}$$

$$H_4 = \sum_{j\in\Lambda} [f^{abp} f^{pcd} t_j^a t_{j+1}^b t_{j+2}^c t_{j+3}^d + t_j^a t_{j+2}^a]. \tag{7.5}$$

As for $M = 2$, we can define the spin polynomials $f_n(\mathcal{C})$, for an arbitrary cluster $\mathcal{C} = \{j_1, j_2, \ldots, j_n\}$, as follows:

$$f_n(\mathcal{C}) = f^{a_1 a_2 b_2} f^{b_2 a_3 b_3} \ldots f^{b_{n-3} a_{n-2} b_{n-2}} f^{b_{n-2} a_{n-1} a_n} t_{j_1}^{a_1} t_{j_2}^{a_2} \ldots t_{j_n}^{a_n}. \tag{7.6}$$

The general expression for the $su(M)$ XXX conserved charges can then be written in a universal form, (corresponding to the Catalan tree in Fig. 1.), valid for all M:

$$H_n = F_{n,0} + \sum_{k=1}^{[n/2]-1} \sum_{\ell=1}^{k} \alpha_{k,\ell} F_{n-2k,\ell}, \tag{7.7}$$

with

$$F_{n,k} = \sum_{\mathcal{C}\in\mathcal{C}^{(n,k)}} f_n(\mathcal{C}). \tag{7.8}$$

The proof that the H_n's form a commuting family parallels the analogous proof in the $su(2)$ case. First, we show that the identities (4.8)-(4.10) remain valid for $su(M)$, but now with $\mathbf{A} = \{t_j^1, \ldots, t_j^{M^2-1}\}$ and $\mathbf{B} = \{t_k^1, \ldots, t_k^{M^2-1}\}$, (with $j \neq k$) and with $\mathbf{A} \times \mathbf{B}$ understood as a vector with $M^2 - 1$ components given by

$$(\mathbf{A} \times \mathbf{B})^c = f^{abc} A^a B^b. \tag{7.9}$$

The demonstration of (4.8)-(4.10) is elementary and uses several identities for the tensors f^{abc} and d^{abc}:[16]

$$\begin{aligned} f^{abc}f^{cde}f^{ega} &= -(M/2)f^{bdg}, \\ f^{abc}f^{cde}d^{ega} &= -(M/2)d^{bdg}, \\ f^{abc}f^{abd} &= M\delta^{cd}, \\ f^{abc}f^{cde}f^{egh}d^{hai} &= -f^{adc}f^{cbe}f^{eih}d^{hag}. \end{aligned} \tag{7.10}$$

The rest of the proof is exactly the same as in the $su(2)$ case: from (4.8)-(4.10) one derives (4.16) and proves first $[H_2, H_n] = 0$ and then the mutual commutativity of all the H_n's.

7.3. Relation to higher spin su(2) models

The $su(M)$ XXX system can be also regarded as a $su(2)$ spin-s chain, with $s = (M-1)/2$. The hamiltonian (7.1) can be rewritten in terms of the variables S^a_j, which are the spin-s $su(2)$ generators, acting non-trivially only at site j:

$$H = \sum_{j\in\Lambda} G_M(x_j), \tag{7.11}$$

where G_M is a polynomial of order M^2-1, and $x_j = S^a_j S^a_{j+1}$. In particular, the expressions for the first few polynomials G_M (obtained by expressing the matrices t^a in terms of the standard spin-s matrices) read:

$$\begin{aligned} G_3(x) =& x + x^2, \\ G_4(x) =& x - \frac{4}{81}(11x^2 + 4x^3), \\ G_5(x) =& x + \frac{1}{90}(13x^2 - 6x^3 - x^4), \\ G_6(x) =& x + \frac{8}{178435}(14911x^2 + 636x^3 - 360x^4 - 32x^5), \\ G_7(x) =& x + \frac{1}{14094}(-6417x^2 - 1713x^3 - 8x^4 + 19x^5 + x^6). \end{aligned} \tag{7.12}$$

[16] The proofs of the first three identities can be found in [39]. The fourth identity can be proved using the first two equalities as well as two additional well known identities:

$$\begin{aligned} f^{adc}f^{cbe} + f^{bdc}f^{ace} + f^{edc}f^{abc} = 0, \\ f^{adc}d^{cbe} + f^{bdc}d^{ace} + f^{edc}d^{abc} = 0. \end{aligned}$$

It is convenient to use the basis provided by the projection operators P_s^j, defined by

$$P_s^j(x) = \prod_{\substack{l=0 \\ l\neq j}}^{2s} \frac{x - y_j}{y_l - y_j}, \tag{7.13}$$

where $j = 0, \dots 2s$, and

$$y_l = \tfrac{1}{2}[2l(l+1) - s(s+1)]. \tag{7.14}$$

Acting on the states of the tensor product of two spins, P_s^j projects onto the states with total spin j. In terms of this basis, modulo constants,

$$G_M(x) = \sum_{j=0}^{2s} (-1)^j P^j_{(M-1)/2}(x). \tag{7.15}$$

Similarly, the density of a XXX conserved charge of order n in terms of the $su(2)$ variables becomes a polynomial in n (matrix) variables $x_j, \dots, x_{j+n-1}$.

8. Equivalent representations of charges in spin chains

8.1. $su(2)$ spin chains in the Weyl representation

It is well known that the $su(2)$ generators can be represented as differential operators acting in the space of functions of two complex variables u, v satisfying $|u|^2 + |v|^2 = 1$:

$$\begin{aligned} \sigma^z &= u\partial_u - v\partial_v, \\ \sigma^x &= u\partial_v + v\partial_u, \\ \sigma^y &= i(v\partial_u - u\partial_v). \end{aligned} \tag{8.1}$$

This is often called the Weyl representation. The spin-s representation can be obtained by the restriction of the space of functions to polynomials of degree $2s$. States of the spin-s chain with N sites can be therefore represented by polynomials of degree $2s$ in $2N$ variables u_i, v_i $(i = 1, \dots, N)$. The conserved charges H_n become commuting differential operators of degree $2n$ in the variables $u_i, v_i, \partial_{u_i}, \partial_{v_i}$. For example, the XXZ hamiltonian in the Weyl representation is:

$$\begin{aligned} H_2 = \sum_{j\in\Lambda} [&(2-\lambda_z)(u_j v_{j+1}\partial_{v_j}\partial_{u_{j+1}} + v_j u_{j+1}\partial_{u_j}\partial_{v_{j+1}}) \\ &+ \lambda_z(u_j u_{j+1}\partial_{u_j}\partial_{u_{j+1}} + v_j v_{j+1}\partial_{v_j}\partial_{v_{j+1}})]. \end{aligned} \tag{8.2}$$

Higher charges can be obtained in the same manner, but the resulting expressions do not seem particularly transparent.

Observe that in deriving the explicit form (4.2) of the charges H_n for the $s = 1/2$ XXX chain one makes use of both the $su(2)$ commutator and the anticommutator $\sigma^a\sigma^b + \sigma^b\sigma^a = \delta^{ab}$. In the Weyl representation, this anticommutator is not encoded in (8.1), but results from the restriction of the representation space to multilinear polynomials. It is therefore clear that the charges (4.2) cease to commute for spin chains with $s > 1/2$. Establishing the explicit form of the charges in this case is a much more difficult task.

8.2. Representations of the XXX charges in terms of braids

For both the $s = 1/2$ and the $su(M)$ XXX models, the hamiltonian can be written, modulo constants, as:

$$H = \sum_{j\in\Lambda} P_{j,j+1}, \tag{8.3}$$

where $P_{j,k}$ are operators exchanging particles at sites j and k. The spin polynomials $f_n(C)$, where $\mathcal{C} = \{i_1, i_2, \ldots, i_{n-1}, i_n\}$ can be rewritten in terms of the exchange operators as follows:

$$f_n(C) = [...[P_{i_1,i_2}, P_{i_2,i_3}], P_{i_3,i_4}], P_{i_4,i_5}], ..., P_{i_{n-1},i_n}]. \tag{8.4}$$

This permits to express the higher-order hamiltonians for the XXX system as a sum of products of the exchange operators.

Let $b_i = P_{i,i+1}$ be a nearest-neighbor exchange operator. It is simple to check that the b_i's satisfy the defining algebra of the braid group B_N, (where $N = |\Lambda|$):

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}, \\ b_i b_j &= b_j b_i, \quad |i-j| > 1, \end{aligned} \tag{8.5}$$

and an additional constraint

$$b_i^2 = 1. \tag{8.6}$$

Note that any binary exchange (not necessarily nearest-neighbor) can be realized as a product of nearest-neighbor exchanges. Therefore, it follows that the spin polynomials (8.4) can be expressed in terms of the generators of B_N. For example, if $\mathcal{C}$ contains only adjacent spins,

$$f_n(C) = [...[b_1, b_2], b_3], b_4], ..., b_n]. \tag{8.7}$$

Hence the XXX charges can be interpreted in terms of equivalence classes (with respect to (8.6)) of braid polynomials (see Fig. 4. for an example).

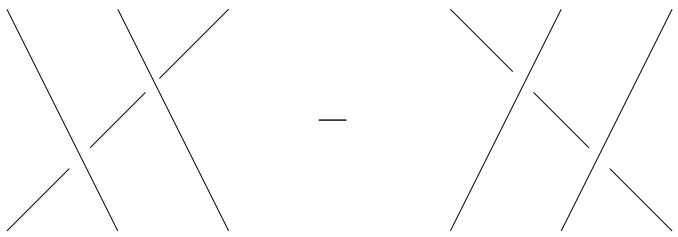

Fig. 4. A braid polynomial corresponding to the three-spin XXX charge

9. Structure of conserved charges in the Hubbard model

9.1. Introduction

The Hubbard model (for which original references can be found in [40]) is usually defined in terms of fermionic operators, by the hamiltonian

$$H = -2 \sum_{j,\ s=\uparrow,\downarrow} (a^\dagger_{j,s} a_{j+1,s} + a^\dagger_{j+1,s} a_{j,s}) + 4U \sum_{j\in\Lambda} (n_{j,\uparrow} - 1/2)(n_{j,\downarrow} - 1/2), \tag{9.1}$$

where

$$n_{j,s} = a^\dagger_{j,s} a_{j,s}, \tag{9.2}$$

and U is some coupling constant. The sum over j runs from 1 to N, the number of sites. The operators $a^\dagger_{j,s}$ and $a_{j,s}$ are respectively the creation and the annihilation operator of an electron of spin s at site j, subject to the anti-commutation relation:

$$a^\dagger_{j,s} a_{k,s'} + a_{k,s'} a^\dagger_{j,s} = \delta_{j,k} \delta_{s,s'}, \tag{9.3}$$

and

$$a^\dagger_{j,s} a^\dagger_{k,s'} = -a^\dagger_{k,s'} a^\dagger_{j,s}, \qquad a_{j,s} a_{k,s'} = -a_{k,s'} a_{j,s}. \tag{9.4}$$

The hamiltonian can be rewritten in terms of spin variables by means of the standard Jordan-Wigner transformation:

$$\begin{aligned} a_{j,\uparrow} &= (\sigma_1^z \sigma_2^z ... \sigma_{j-1}^z)\ \sigma_j^-, \\ a_{j,\downarrow} &= (\sigma_1^z \sigma_2^z ... \sigma_N^z)(\tau_1^z \tau_2^z ... \tau_{j-1}^z)\ \tau_j^-. \end{aligned} \tag{9.5}$$

$\{\sigma_j^a\}$ and $\{\tau_j^a\}$, $a \in \{x, y, x\}$, $j \in \Lambda$, are two independent sets of Pauli matrices:

$$[\sigma_j^i, \tau_k^\ell] = 0, \tag{9.6}$$

and the normalization used for $\sigma_j^\pm$ is

$$\sigma_j^\pm = \tfrac{1}{2}(\sigma_j^x \pm i\sigma_j^y). \tag{9.7}$$

In this way, the hamiltonian $H = H_2$ is easily seen to be equivalent to (1.2). Notice that for $U = 0$, the Hubbard model reduces to two uncoupled XX models and that for $\tau_j^\pm = 0$ and $\tau_j^z = h$ for all j, it reduces to the XX model in an external magnetic field.

The first non-trivial conservation law was obtained by Shastry and found to be [6]:

$$\begin{aligned} H_3 = \sum_{j\in\Lambda} [&(\sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^y - \sigma_j^y \sigma_{j+1}^z \sigma_{j+2}^x) \\ &- U(\sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x)(\tau_j^z + \tau_{j+1}^z)] + [\sigma \leftrightarrow \tau]. \end{aligned} \tag{9.8}$$

It is simple to verify that it indeed commutes with H_2.

To conclude this short introduction of the Hubbard model, we point out that there exists another equivalent way to represent it: redefining the spin variables as $\sigma_j^a \to S_{2j}^a$ and $\tau_j^a \to S_{2j+1}^a$, we obtain a lattice with $2N$ sites with next-to-nearest neighbor interactions and bond alternations:

$$H_2 = \sum_j S_j^+ S_{j+2}^- + S_j^- S_{j+2}^+ + U[1 + (-)^j] S_j^z S_{j+1}^z. \tag{9.9}$$

9.2. The non-existence of a ladder operator

We first demonstrate that for the Hubbard model there is no ladder operator B whose commutator with H_2 would produce H_3. The most general possible expression for B is [21]:

$$B = \sum_{j\in\Lambda} [\ jb_j + c_i], \tag{9.10}$$

where b_j and c_j are translation invariant densities, at most bilinear in (nearest-neighbor) spin variables. Enforcing the symmetry $\sigma \leftrightarrow \tau$ and requiring B to reproduce the known ladder operator in the two limiting cases $U \to 0$ and $\tau_j^z \to h$ gives

$$\begin{aligned} b_j =& \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \tau_j^x \tau_{j+1}^x + \tau_j^y \tau_{j+1}^y + \frac{U}{2}(\sigma_j^z \tau_j^z + \sigma_{j+1}^z \tau_{j+1}^z), \\ c_j =& 0. \end{aligned} \tag{9.11}$$

The resulting B is thus the first moment of the hamiltonian, with the density of the latter symmetrized with respect to sites j and $j+1$:

$$B = \sum_{j \in \Lambda} j[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \tau_j^x \tau_{j+1}^x + \tau_j^y \tau_{j+1}^y + \frac{U}{2}(\sigma_j^z \tau_j^z + \sigma_{j+1}^z \tau_{j+1}^z)]. \tag{9.12}$$

This form of B is in fact generic to all the cases where it exists.

With this candidate ladder operator, a simple calculation yields:

$$\begin{aligned} [B, H_2] = -2i \sum_{j \in \Lambda} [& \sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^y - \sigma_j^y \sigma_{j+1}^z \sigma_{j+2}^x \\ & - \frac{U}{2}(\sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x)(\tau_{j+1}^z + \tau_j^z)] + [\sigma \leftrightarrow \tau]. \end{aligned} \tag{9.13}$$

The sum has the same form as the above H_3, except that the coefficient of the U term is here $-1/2$ instead of -1. This particular value of the relative coefficient between the U independent term and the linear one is of course crucial for the commutativity of H_3 with H_2. In other words, $[H_2, [B, H_2]]$ does not vanish. This finishes the proof of the non-existence of a ladder operator for the Hubbard model.

9.3. Higher order charges

As already indicated in the introduction, higher order charges must be calculated by

brute force methods and they become rather complicated. For instance H_4 reads:[17]

$$\begin{aligned}
H_4 =&\sum_{j\in\Lambda}[(\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^z\sigma_{j+3}^x+\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^z\sigma_{j+3}^y)\\
&-U(\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x+\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^y)(\tau_j^z+\tau_{j+1}^z+\tau_{j+2}^z)\\
&-\frac{U}{2}(\sigma_j^x\sigma_{j+1}^y-\sigma_j^y\sigma_{j+1}^x)(\tau_j^x\tau_{j+1}^y-\tau_j^y\tau_{j+1}^x)\\
&-U(\sigma_j^x\sigma_{j+1}^y-\sigma_j^y\sigma_{j+1}^x)(\tau_{j+1}^x\tau_{j+2}^y-\tau_{j+1}^y\tau_{j+2}^x)\\
&-U(\sigma_j^z\tau_{j+1}^z)-\frac{U}{2}(\sigma_j^z\tau_j^z)\\
&+U^2(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y)(\tau_j^z\tau_{j+1}^z+1)]\\
&+[\sigma\leftrightarrow\tau].
\end{aligned}\tag{9.14}$$

This charge has also been found in [41, 19]. Observe that these integrals of motion can be written much more compactly in terms of the tensor products of the densities of the XX charges. It is convenient to use a notation which differs slightly from the one used in section (6); namely, we define:

$$\begin{aligned}
h_{n,j}^{(+)} =&e_{n,j}^{xx}+e_{n,j}^{yy}=\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^z...\sigma_{j+n-1}^z\sigma_{j+n}^x+\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^z...\sigma_{j+n-1}^z\sigma_{j+n}^y,\\
h_{n,j}^{(-)} =&e_{n,j}^{xy}-e_{n,j}^{yx}=\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^z...\sigma_{j+n-1}^z\sigma_{j+n}^y-\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^z...\sigma_{j+n-1}^z\sigma_{j+n}^x,
\end{aligned}\tag{9.15}$$

for $n>1$, and

$$h_{1,j}^{(+)}=-\sigma_j^z\ ,\qquad h_{1,j}^{(-)}=0,\tag{9.16}$$

and similarly

$$g_{n,j}^{(\pm)}(\tau)=h_{n,j}^{(\pm)}(\sigma\to\tau).\tag{9.17}$$

With this notation, the first three charges take the form:

$$\begin{aligned}
H_2 =&\sum_{j\in\Lambda}[h_{2,j}^{(+)}+g_{2,j}^{(+)}+U\ h_{1,j}^{(+)}g_{1,j}^{(+)}],\\
H_3 =&\sum_{j\in\Lambda}[h_{3,j}^{(-)}+g_{3,j}^{(-)}+U\ \{h_{2,j}^{(-)}(g_{1,j}^{(+)}+g_{1,j+1}^{(+)})+g_{2,j}^{(-)}(h_{1,j}^{(+)}+h_{1,j+1}^{(+)})\}],\\
H_4 =&\sum_{j\in\Lambda}[h_{4,j}^{(+)}+g_{4,j}^{(+)}+U\ \{h_{3,j}^{(+)}(g_{1,j}^{(+)}+g_{1,j+1}^{(+)}+g_{1,j+2}^{(+)}),\\
&+g_{3,j}^{(+)}(h_{1,j}^{(+)}+h_{1,j+1}^{(+)}+h_{1,j+2}^{(+)})-h_{2,j}^{(-)}(g_{2,j-1}^{(-)}+g_{2,j}^{(-)}+g_{2,j+1}^{(-)})\\
&-h_{1,j}^{(+)}(g_{1,j-1}^{(+)}+g_{1,j}^{(+)}+g_{1,j+1}^{(+)})\}\\
&+U^2\ \{h_{2,j}^{(+)}(g_{1,j}^{(+)}g_{1,j+1}^{(+)}+1)+g_{2,j}^{(+)}(h_{1,j}^{(+)}h_{1,j+1}^{(+)}+1)\}].
\end{aligned}\tag{9.18}$$

[17] One could have hoped that the above operator B would still be of some use, at least for generating the correct terms in H_4 (despite the fact that the coefficients cannot be expected to be right) in the commutator $[B,H_3]$. But some terms in H_4 are not produced in this way.

We have also obtained the explicit form of H_5 which is

$$\begin{aligned} H_5 =&\sum_{j\in\Lambda}[h^{(-)}_{5,j} + U\ \{h^{(-)}_{4,j}(g^{(+)}_{1,j} + g^{(+)}_{1,j+1} + g^{(+)}_{1,j+2} + g^{(+)}_{1,j+3}) \\ &+ h^{(+)}_{3,j}(g^{(-)}_{2,j-1} + g^{(-)}_{2,j} + g^{(-)}_{2,j+1} + g^{(-)}_{2,j+2}) \\ &- h^{(-)}_{2,j}(g^{(+)}_{1,j-1} + g^{(+)}_{1,j} + g^{(+)}_{1,j+1} + g^{(+)}_{1,j+2})\} \\ &+ U^2\ \{h^{(-)}_{3,j}(g^{(+)}_{1,j}g^{(+)}_{1,j+1} + g^{(+)}_{1,j}g^{(+)}_{1,j+2} + g^{(+)}_{1,j+1}g^{(+)}_{1,j+2}) \\ &+ h^{(+)}_{2,j}(g^{(-)}_{2,j-1}g^{(+)}_{1,j+1} + g^{(-)}_{2,j+1}g^{(+)}_{1,j})\} \\ &- U^3\ \{h^{(-)}_{2,j}(g^{(+)}_{1,j} + g^{(+)}_{1,j+1})\}] + [h \leftrightarrow g]. \end{aligned} \tag{9.19}$$

We now present a number of simple observations concerning the structure of the conserved charges, indicated by the above results.

1- H_n is composed of sum of terms of the form

$$h^{(\epsilon)}_{\ell_1}h^{(\epsilon')}_{\ell_2}...g^{(\epsilon'')}_{\ell_{p-1}}g^{(\epsilon''')}_{\ell_p}, \tag{9.20}$$

with $\ell_i \geq 1$, $i = 1, \ldots, p$, where p is some integer and $\epsilon, \epsilon', ... = \pm$. A parity of (9.20) is given by

$$\epsilon\epsilon'...\epsilon''\epsilon''' = (-)^n. \tag{9.21}$$

Furthermore, these terms must all be scalars, which means that in the product (9.20) pseudoscalar factors must occur in pairs.[18]

2- With the normalization we have chosen for H_2, the relative coefficients of the various components of H_n are $\pm$ times some powers of U.

3- The term independent of U in H_n is given by

$$H_n(U = 0) = \sum_{j\in\Lambda}[h^{(\epsilon)}_{n,j} + g^{(\epsilon)}_{n,j}], \tag{9.22}$$

with $\epsilon = (-)^n$. Terms depending upon U involve products of h and g factors of the form (9.20) with

$$\sum_{i=1}^{p}\ell_i = n - 2r, \tag{9.23}$$

[18] A pseudoscalar changes sign when the direction of the z-axis is reversed. $h^{(+)}_n$ is scalar (pseudoscalar) for n even (odd), while $h^{(-)}_n$ is scalar (pseudoscalar) for n odd (even). The same holds for g.

where r is a non-negative integer. These terms can be therefore grouped into classes, associated to the partitions of $n-2r$, for $r=0,\ldots,[n/2]-1$.
4- The terms in H_n associated to partitions of n (those for which $r=0$) have a simple pattern. The number of parts p in the partition is related to the power of U in the coefficient of the corresponding term in H_n. This power is simply $p-1$. For example, the terms with $r=0$ which are linear in U are obtained from all possible decomposition of n into two parts of distinct 'colors' (the σ and τ factors), e.g.

$$5=4+(1)=1+(4)=3+(2)=2+(3), \tag{9.24}$$

(where terms referring to g factors are written in parentheses). Similarly the terms of order U^2 are associated with partitions of n into three parts of two colors. For instance, the terms appearing with a coefficient of absolute value U^2 in H_5 are in correspondence with the partitions

$$5=3+(1+1)=1+1+(3)=2+(2+1)=2+1+(2). \tag{9.25}$$

But not all partitions into more than two parts are eligible (e.g., the partition $5=(1)+2+2$ is absent). Some, but not all, are eliminated by the scalar and/or the parity conditions stated above. For instance, $5=1+1+1+1+1$, with any arrangement of parentheses, is excluded by the parity and the scalar condition.
5- The above correspondence is not one-to-one since more than one term is associated to a given partition. Consider a class of terms corresponding to a certain partition. Terms within such class are related to each other by a relative translation of their components. For the first two-component partition, i.e., $n=n-1+(1)$, the different terms in the class correspond to the $n-1$ positions of $g^{(+)}_{1,k}$, $k=j,...,j+n-1$, relative to the $h_{n-1,j}$ factor. For the other two-component partitions, there are also $n-1$ terms in the class, with the difference that now the first term in the class $n=n-k+(k)$ is $g_{k,j-k+1}$ and the other terms are obtained by translating this g factor by one unit with respect to $h_{n-k,j}$, up to the site $j+n-k$. The different terms in classes labeled by partitions into more parts can be described similarly.

It is more difficult to find a simple pattern for the terms associated to partitions of $n-2r,\ r>1$ because they are not universal, in the sense that they can be modified by the addition of a linear combination of the lower order charges H_{n-2r}. Nevertheless, the terms linear in U are described by the explicit formula presented below in section (9.5).

9.4. A diagrammatic description of the conserved charges

Formulated in words, some of the above rules may look a bit complicated. However they have a very simple diagrammatic description. To the densities $h^{(\pm)}_{n,j}$, $g^{(\pm)}_{n,j}$, we will assign sequences of n symbols (corresponding to n successive sites of the chain starting at site j), as follows:

$$\begin{aligned} h^{(+)}_{n,j} &= \oplus\oplus\ldots\oplus\oplus, \\ h^{(-)}_{n,j} &= \ominus\ominus\ldots\ominus\ominus, \\ g^{(+)}_{n,j} &= \otimes\otimes\ldots\otimes\otimes, \\ g^{(-)}_{n,j} &= \oslash\oslash\ldots\oslash\oslash. \end{aligned}$$

The product of two or more densities will be represented by writing the corresponding sequences of symbols, appropriately shifted, in successive rows, e.g:

$$h^{(-)}_{2,j}\, g^{(+)}_{2,j+1} = \begin{matrix} \ominus\ominus \\ \otimes\otimes \end{matrix} \quad .$$

It is understood that the leftmost symbol in the top row is at site j. In this notation, (omiting the sums over j), the charges read:

$$H_2 = \oplus\oplus \;+\; \otimes\otimes \;+\; U\begin{matrix}\oplus\\ \otimes\end{matrix},$$

$$H_3 = \ominus\ominus\ominus \;+\; U(\begin{matrix}\ominus\ominus\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\\ \otimes\end{matrix}) + (\sigma \leftrightarrow \tau),$$

$$\begin{aligned} H_4 = {} & \oplus\oplus\oplus\oplus \;+\; U(\begin{matrix}\oplus\oplus\oplus\\ \otimes\end{matrix} + \begin{matrix}\oplus\oplus\oplus\\ \otimes\end{matrix} + \begin{matrix}\oplus\oplus\oplus\\ \otimes\end{matrix}) \\ & + U(\begin{matrix}\oplus\\ \otimes\otimes\otimes\end{matrix} + \begin{matrix}\oplus\\ \otimes\otimes\otimes\end{matrix} + \begin{matrix}\oplus\\ \otimes\otimes\otimes\end{matrix}) \\ & - U(\begin{matrix}\ominus\ominus\\ \oslash\oslash\end{matrix} + \begin{matrix}\ominus\ominus\\ \oslash\oslash\end{matrix} + \begin{matrix}\ominus\ominus\\ \oslash\oslash\end{matrix}) \\ & - U\,(\begin{matrix}\oplus\\ \otimes\end{matrix} + \begin{matrix}\oplus\\ \otimes\end{matrix} + \begin{matrix}\oplus\\ \otimes\end{matrix}) + U^2(\begin{matrix}\oplus\oplus\\ \otimes\\ \otimes\end{matrix}) + U^2(\begin{matrix}\oplus\\ \oplus\\ \otimes\otimes\end{matrix}) \;-\; U^3(\begin{matrix}\oplus\\ \otimes\end{matrix}), \end{aligned}$$

$$\begin{aligned} H_5 = {} & \ominus\ominus\ominus\ominus\ominus \;+\; U\,(\begin{matrix}\ominus\ominus\ominus\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\ominus\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\ominus\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\ominus\\ \otimes\end{matrix}) \\ & + U\,(\begin{matrix}\oplus\oplus\oplus\\ \oslash\oslash\end{matrix} + \begin{matrix}\oplus\oplus\oplus\\ \oslash\oslash\end{matrix} + \begin{matrix}\oplus\oplus\oplus\\ \oslash\oslash\end{matrix} + \begin{matrix}\oplus\oplus\oplus\\ \oslash\oslash\end{matrix}) \\ & - U\,(\begin{matrix}\ominus\ominus\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\\ \otimes\end{matrix}) \\ & + U^2\,(\begin{matrix}\ominus\ominus\ominus\\ \otimes\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\ominus\\ \otimes\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\ominus\\ \otimes\\ \otimes\end{matrix}) \;+\; U^2\,(\begin{matrix}\oplus\oplus\\ \oslash\oslash\\ \otimes\end{matrix} + \begin{matrix}\oplus\oplus\\ \oslash\oslash\\ \otimes\end{matrix}) \\ & - U^3\,(\begin{matrix}\ominus\ominus\\ \otimes\end{matrix} + \begin{matrix}\ominus\ominus\\ \otimes\end{matrix}) \;+\; (\sigma \leftrightarrow \tau). \end{aligned}$$

9.5. The explicit form of the term linear in U

We now give the explicit expression of the term linear in U in the charge H_n:

$$\begin{aligned} H_n = \sum_{j\in\Lambda} [& h_{n,j}^{(-)^n} + g_{n,j}^{(-)^n} \\ & + U \sum_{k=0}^{[n/2]-1} \sum_{m=1}^{n-2k-1} \sum_{\ell=0}^{n-2} (-)^{n+k+m(n-m)+1}\ h_{n-m-2k,j}^{(-)^{n+m+1}}\ g_{m,j-m-k+\ell+1}^{(-)^{m+1}}] \\ & + \mathcal{O}(U^2) \\ \equiv\ & H_n^{(0)} + U\ H_n^{(1)} + \mathcal{O}(U^2). \end{aligned} \tag{9.26}$$

We now explain the origin of the three internal summations for the linear term. The first one, over k, keeps track of all the integers $n-2k$ whose two-term partitions are to be considered. The second summation takes care of all the possible ways the integer $n-2k$ can be separated into two parts. Finally, the sum over ℓ generates the different terms of a given class, that is the different translations of the g factor with respect to h, whose position is kept fixed. Although it may not be manifest, this expression is symmetric with respect to the interchange of h and g.

To prove this result we have to show that, up to terms of order U^2, (9.26) commutes with H_2. The proof is based on the following general commutation relations:

$$\begin{aligned} [H_2^{(0)}, h_{n,j}^{(\pm)}] &= [\sum_{i\in\Lambda} h_{2,i}^{(+)}, h_{n,j}^{(\pm)}] \\ &= \pm 2i\{h_{n+1,j-1}^{(\mp)} + h_{n+1,j}^{(\mp)} + \delta(h_{n-1,j+1}^{(\mp)} - h_{n-1,j}^{(\mp)})\}, \end{aligned} \tag{9.27}$$

and

$$\begin{aligned} [H_2^{(1)}, h_{n,j}^{(\pm)}] &= [\sum_{i\in\Lambda} h_{1,i}^{(+)} g_{1,i}^{(+)}, h_{n,j}^{(\pm)}] \\ &= \pm 2i\{h_{n,j}^{(\mp)} g_{1,j}^{(+)} - h_{n,j}^{(\mp)} g_{1,j+n-1}^{(+)}\}. \end{aligned} \tag{9.28}$$

In the first expression, we have introduced an operator δ whose action on $h_{n,i}^{(\pm)}$ or $g_{n,i}^{(\pm)}$ is defined to be multiplication by 2 if $n=1$ and 1 otherwise:

$$\delta h_{1,j}^{(+)} = 2h_{1,j}^{(+)}\ , \qquad \delta h_{n\neq 1,j}^{(\pm)} = h_{n,j}^{(\pm)}. \tag{9.29}$$

Exactly the same results applies for $h \to g$. Notice that both type of commutators change the parity of the density on which it acts.

These commutators are very easily demonstrated using the definitions (9.15), (9.16) and (9.17).

We now proceed to the evaluation of $[H_2^{(0)}, H_n^{(1)}]$. Using (9.27), one gets

$$
\begin{aligned}
[H_2^{(0)}, H_n^{(1)}] = 2i \sum_{j\in\Lambda} \sum_{k=0}^{[n/2]-1} \sum_{m=1}^{n-2k-1} \sum_{\ell=0}^{n-2} (-)^{n+k+m(n-m)+1} \\
\times \{(-)^{n+m+1}[h^{(-)^{n+m}}_{n-m-2k+1,j-1} - h^{(-)^{n+m}}_{n-m-2k+1,j} + \delta h^{(-)^{n+m}}_{n-m-2k-1,j+1} \\
- \delta h^{(-)^{n+m}}_{n-m-2k-1,j}]\; g^{(-)^{m+1}}_{m,j-m-k+\ell+1} \\
+ (-)^{m+1} h^{(-)^{n+m+1}}_{n-m-2k,j}[g^{(-)^{m}}_{m+1,j-m-k+\ell} - g^{(-)^{m}}_{m+1,j-m-k+\ell+1} \\
+ \delta g^{(-)^{m}}_{m-1,j-m-k+\ell+2} - \delta g^{(-)^{m}}_{m-1,j-m-k+\ell+1}]\}.
\end{aligned}
\tag{9.30}
$$

We first relabel the j summation index in each term in order to have all h factors at site j (e.g. $j-1 \to j$ in the first term, etc.). Next, in the first, second, seventh and eighth terms, we relabel m as: $m-1 \to m$. We then evaluate the summation over ℓ. All terms cancel two by two except for those at the boundaries of the ℓ interval. For the summation over m, a similar situation holds, except that now two values at both extremities of the interval contribute due to the δ factors which are responsible for partial cancellations. Most of the resulting terms can again be canceled if in half of them we reshuffle the k index as: $k+1 \to k$. Summing over k yields then

$$
[H_2^{(0)}, H_n^{(1)}] = 2i \sum_{j\in\Lambda} (-)^{n+1} \{h^{(-)^{n+1}}_{n,j} (g^{(+)}_{1,j} - g^{(+)}_{1,j+n-1}) + g^{(-)^{n+1}}_{n,j} (h^{(+)}_{1,j} - h^{(+)}_{1,j+n-1})\}, \tag{9.31}
$$

which is exactly the same as $-[H_2^{(1)}, H_n^{(0)}]$. This shows that, up to terms of order U^2, $[H_2, H_n]$ indeed vanishes.

10. Concluding remarks

The main concern of the present work is to exhibit the algebraic structure of conservation laws in integrable spin chains. For the XYZ model, this has led to a description in terms of simple multilinear polynomials in spin variables. The terms describing a charge H_n can be grouped into classes corresponding to different types of clusters with prescribed number of spins and holes. The coefficients of the different terms satisfy recursion relations. In the general anisotropic case, we have not been able to solve these relations. In two special cases however, for the XXX and XY models, explicit expressions for all the conserved charges can be found in closed form. The structure of charges that emerges for the Hubbard model is organized in terms of tensor products of densities of two XX models.

However, in the absence of a recursive structure we were able to prove this only for the zero-order and linear terms in the model's free parameter.

For the XXX and XY models, the explicit construction of the charges provides an alternative proof of integrability, independent of the Bethe Ansatz or the transfer matrix formalism. Admittedly, this pedestrian proof is quite tedious and it does not even adress the issue of finding the eigenstates and eigenvalues of the hamiltonian. On the other hand, the proof for the Heisenberg model can be generalized in a completely straightforward way for the $su(M)$ invariant XXX chain (which required the use of the nested Bethe Ansatz). One may therefore hope that there are other types of systems in which the unorthodox direct approach could be effectively used to prove integrability.

One expects to find a similar pattern of conserved charges in integrable chains with non-trivial boundary conditions. In particular, for the XYZ model with nonzero boundary terms in the hamiltonian, the construction of higher order conserved charges can be achieved by modifying the densities of the XYZ higher order charges only near the boundary; in the "bulk," the densities of the conserved charges are not affected by boundary effects. A similar reasoning can be made for integrable chains with impurities. It is also of interest to find explicitly the deformation of higher order conserved charges in integrable models with a general quantum group symmetry.

Another very interesting problem is to determine the structure of conservation laws in integrable chains with long-range interactions. The explicit expressions for the charges H_3 and H_4 in the Haldane-Shastry model [42] are analogous to the corresponding formulae for the Heisenberg chain. Indeed, one can regard the Heisenberg model as a limiting case of a long-range model. It is thus natural to expect the structure of the charges in the models of the Haldane-Shastry type to be similar to the XXX case.

In a different perspective, the structure of the higher conserved charges for the models analyzed here may provide new insights into the problem of testing integrability for general spin chains. The integrability of a quantum chain is usually demonstrated rather indirectly by showing that the model can be solved by the coordinate Bethe ansatz, or that the hamiltonian can be derived from a commuting family of transfer matrices related to the Yang-Baxter equation. But these are only sufficient conditions for integrability. Moreover, testing these sufficient conditions is often not easy and systematic. Alternatively, one may look directly at the existence of higher order charges, with a general structure similar as in the basic models considered in this work. A heuristic integrability test based on this approach will be considered in a forthcoming publication.

For higher spin chains, anisotropic $su(M)$ chains, or models with underlying algebra different from $su(M)$, the determination of the explicit form of the conservation laws remains a difficult challenge. As we have stressed in the introduction, the problem is not only with the actual computation - here one may be helped, to a certain extent, by effective computer algebra programs - but in finding a pattern in the huge amounts of data that emerge out of these calculations. In our view, integrability reflects itself into such structural patterns.

Appendix A. Mastersymmetries and hamiltonian structures in classical soliton theory

A.1. Symmetries and mastersymmetries

We consider continuous systems subject to the evolution equation

$$\varphi_t = K_1(\varphi), \tag{A.1}$$

where φ collectively describes the independent fields under consideration. K_1 stands for a vector field whose evaluation at the point φ yields the scalar quantity $K_1(\varphi)$. The Lie derivative in the direction of K_1, noted L_{K_1}, is the same as the time derivative along the integral curves of the above evolution equation. Thus, the time derivative of a generic tensor field A, that can depend explicitly upon time, is simply $\partial_t A + L_{K_1} A$. A is said to be invariant with respect to (A.1) if its time derivative vanishes,

$$\partial_t A + L_{K_1} A = 0. \tag{A.2}$$

A vector field satisfying (A.2) is called a symmetry of (A.1). The Lie derivative of a vector field is defined as

$$L_K A \equiv [K, A]_L = A'[K] - K'[A], \tag{A.3}$$

where $K'(\varphi)[A]$ denotes the Fréchet derivative of K at the point φ in the direction of A:

$$K'[A] = \frac{\partial}{\partial \epsilon} K(\varphi + \epsilon A)|_{\epsilon=0}. \tag{A.4}$$

If (A.1) is one member of an integrable hierarchy, then there exists an infinite number of time independent vector fields K_m all commuting together:

$$[K_n, K_m]_L = 0. \tag{A.5}$$

All $K_{m>1}$ are symmetries of (A.1). A mastersymmetry is defined to be a vector field τ whose commutator with any K_n lies in the commutant of K_n [43, 44, 45] :

$$[[\tau, K_n], K_n]_L = 0, \quad \forall\ n. \tag{A.6}$$

A.2. Hamiltonian structures in integrable systems

The evolution equation (A.1) is said to be hamiltonian [46,47] if it can be written in the form

$$\varphi_t = P\frac{\delta H}{\delta \varphi} = \{\varphi, H\}, \tag{A.7}$$

where the Poisson bracket is defined in terms of the differential operator P by

$$\{\varphi(x), \varphi(y)\} = P(x)\delta(x-y) \tag{A.8}$$

and

$$\frac{\delta H}{\delta \varphi} = \frac{\delta}{\delta \varphi}\int h\ dx = \sum_{k\geq 0}(-\partial_x)^k \frac{\partial h}{\partial(\partial_x^k \varphi)}. \tag{A.9}$$

A (matrix) differential operator P defining a Poisson bracket, satisfying the usual requirements of antisymmetry and the Jacobi identity, is said to be hamiltonian. Take for instance the KdV equation

$$u_t = u_{xxx} + 6uu_x. \tag{A.10}$$

It can be written as a hamiltonian system in two different ways [48]:

$$\begin{aligned} u_t =& P_1\frac{\delta H_3}{\delta u} = \partial_x \frac{\delta}{\delta u}\int (u^3 - \tfrac{1}{2}\ u_x^2)dx \\ =& P_2\frac{\delta H_2}{\delta u} = \tfrac{1}{2}(\partial_x^3 + 4u\partial_x + 2u_x)\frac{\delta}{\delta u}\ \int u^2 dx. \end{aligned} \tag{A.11}$$

We stress that P_1 and P_2 define two distinct Poisson brackets (cf. (1.10)). (Notice also that here the subscript for the conserved charges stands for half the usual degree of the density: deg u = 2 deg ∂_x =2.) Any equation in the KdV hierarchy can be written in the form

$$u_t = P_1\frac{\delta H_{n+1}}{\delta u} = P_2\frac{\delta H_n}{\delta u}. \tag{A.12}$$

This translates into the Lénard scheme (1.11) for calculating the conserved charges. Such a system is said to be bi-hamiltonian[19] and it is characterized to a large extent by its recursion operator

$$R = P_2 P_1^{-1} \tag{A.13}$$

[19] More precisely, it is also required that any linear combination of P_1 and P_2 must be hamiltonian.

(the adjoint of the Lénard recursion operator). It is not difficult to show that if both P_1 and P_2 are hamiltonian, then the infinite sequence of operators

$$P_m \equiv R^{m-2} P_2 \tag{A.14}$$

are also hamiltonian.

A.3. Hamiltonian mastersymmetries for the KdV and NLS equations

A mastersymmetry not only maps (via L_τ) symmetries into new symmetries, but also any invariants (conservation laws, hamiltonian operators, mastersymmetries) into other invariants of the same type. In particular, there is an infinite number of mastersymmetries.

An almost universal symmetry in integrable systems is the scaling symmetry and this can be used as a convenient structural organizing tool for the description of mastersymmetries [44, 45]. Let τ_0 be the time-independent part of the vector field associated with this symmetry (an example is given below). The infinite set of mastersymmetries can be denoted by $\{\tau_n\}$ where in terms of the scaling, τ_n has degree n. The mastersymmetries satisfy the algebra

$$[\tau_n, \tau_m]_L = c(m-n)\tau_{n+m}, \tag{A.15}$$

where $n, m \geq 0$ and c is some constant. We also have

$$[\tau_n, K_m]_L = a_{n,m} K_{n+m} \tag{A.16}$$

(with $a_{n,m} =$ constant). These relations show that the whole structure of the hierarchy can be extracted out of a degree 1 mastersymmetry τ_1. Actually, it is natural to expect that for an evolution equation of the form (A.1), the existence of a degree 1 mastersymmetry is sufficient to ensure the existence of an infinite number of mastersymmetries, a definite characteristic of an integrable system.[20] In terms of τ_1, the conserved charges can be generated recursively by

$$L_{\tau_1} H_m \equiv H'_m[\tau_1] = H_{m+1}. \tag{A.17}$$

[20] This point of view has been mainly advocated by Fuchssteiner. A related conjecture is that integrability is ensured by the existence of one non-Lie point symmetry [45] (section 4 of this reference explains the connection between these concepts). For integrability tests based on symmetries, see [49].

A mastersymmetry is called hamiltonian if it can be written in the form

$$\tau = P\frac{\delta B}{\delta \varphi}, \tag{A.18}$$

for some hamiltonian operator P and some integral B. For bi-hamiltonian systems with scaling symmetry, all mastersymmetries appear to be hamiltonian with respect to an appropriate P_m.

Again, the KdV equation offers the simplest illustration of this feature. We first note that this equation is invariant under the following scaling

$$u(x,t) \to \epsilon^2 u(\epsilon x, \epsilon^3 t). \tag{A.19}$$

This symmetry is generated by the time-dependent vector field

$$\Gamma = 2u + xu_x + 3tu_t, \tag{A.20}$$

whose time-dependent part is the mastersymmetry τ_0. It is not difficult to check that

$$L_{\tau_0} R = 2R. \tag{A.21}$$

An infinite number of mastersymmetries can be generated out of τ_0 by the recursion operator R [44]:

$$\tau_n = R^n \tau_0 = (\partial_x^2 + 4u + 2u_x \partial_x^{-1})^n (2u + xu_x). \tag{A.22}$$

These τ_n satisfy the algebra (A.15) with $c = 2$. In particular

$$\tau_1 = x(u_{xxx} + 6uu_x) + 8u^2 + 4u_{xx} + 2u_x(\partial_x^{-1} u). \tag{A.23}$$

Then, denoting

$$B = \tfrac{1}{2} \int xu \, dx, \tag{A.24}$$

all τ_n are seen to be hamiltonian, each with respect to a different hamiltonian operator [50]:

$$\begin{aligned} \tau_{-1} &= P_1 \frac{\delta B}{\delta u}, \\ \tau_0 &= P_2 \frac{\delta B}{\delta u}, \\ \tau_1 &= P_3 \frac{\delta B}{\delta u}, \\ &\cdots \\ \tau_n &= P_{n+2} \frac{\delta B}{\delta u}. \end{aligned} \tag{A.25}$$

Since

$$L_{\tau_n} H_m = H'_m[\tau_n] = \int \tau_n \frac{\delta H_m}{\delta u}\, dx = H_{n+m}, \tag{A.26}$$

the above relations can be reexpressed in the form

$$\{B, H_n\}_m = H_{n+m-2} \tag{A.27}$$

(where $\{\ ,\ \}_m$ is defined in terms of P_m). This shows that a KdV raising ladder operator (mapping H_n to H_{n+1} through some Poisson bracket) is necessarily defined in terms of a composite hamiltonian operator, namely

$$P_3 = P_2 P_1^{-1} P_2. \tag{A.28}$$

Equivalently, τ_1 is hamiltonian, but with respect to the composite operator P_3. With respect to P_2, B is not a ladder operator. On the other hand, the first moment of the hamiltonian H_2, that is $\frac{1}{2}\ \int xu^2\ dx$, cannot serve as a ladder operator with respect to P_2 because

$$P_2 \frac{\delta}{\delta u} \tfrac{1}{2}\ \int xu^2\ dx = P_2(xu) \tag{A.29}$$

is not a mastersymmetry.

For the nonlinear Schrödinger equation, the analogue of (A.27) also holds, where now

$$B = \int x\phi^*\phi\ dx, \tag{A.30}$$

and the Poisson bracket defined by P_1 is

$$\{\phi^*(x), \phi(y)\} = \delta(x-y), \tag{A.31}$$

whose quantization yields the usual form of the quantum Schrödinger equation. This B acts as a lowering ladder operator:

$$\{B, H_n\}_1 = H_{n-1}. \tag{A.32}$$

For the quantum version of these theories, the implication of these results is clear. The quantum nonlinear Schrödinger equation is obtained from the quantization of the first hamiltonian structure while the quantum KdV equation is obtained by the quantization of the classical KdV second hamiltonian structure. As already stressed in the introduction, for quantum systems the bi-hamiltonian property is lost. The fact that for both cases no classical ladder operator can be defined in terms of the hamiltonian structure appropriate to its quantum version, strongly suggests that no quantum ladder operators exists for the corresponding quantum theories. This conclusion is supported by a direct analysis presented in appendix B.

A.4. Hamiltonian mastersymmetry of the Landau-Lifshitz equation

The situation described above for the KdV case applies to most bi-hamiltonian systems. However, there exist exceptional systems for which the mastersymmetry of degree 1 is hamiltonian with respect to one of the two basic hamiltonian operators $P_{1,2}$. This is exactly what happens for the Landau-Lifshitz equation, to which we now turn.

The canonical equation of motion defined by the XYZ hamiltonian

$$H = \sum_{j\in\Lambda} \lambda_a \sigma_j^a \sigma_{j+1}^a, \tag{A.33}$$

is

$$\frac{d\sigma_j^a}{dt} = [\sigma_j^a, H] = 2i\lambda_b \epsilon_{abc} \sigma_j^c (\sigma_{j-1}^b + \sigma_{j+1}^b). \tag{A.34}$$

Its continuous classical limit is obtained from the substitutions [51]:

$$\begin{aligned} \sigma_i^a &\to S^a(x), & \sigma_{i+1}^a &\to S^a(x+\Delta), \\ \lambda_a &\to 1 + \frac{\Delta^2}{2} J^a, & t &\to -i\Delta^{-2} t, \end{aligned} \tag{A.35}$$

with Δ being the lattice spacing, and it reads

$$\frac{dS^a}{dt} = 2\epsilon_{abc} S^c (J^b S^b + S_{xx}^b). \tag{A.36}$$

(A.36) is called the Landau-Lifshitz equation. It can be written in hamiltonian form by means of the following Poisson bracket

$$\{S^a(x), S^b(y)\} = 2\epsilon_{abc} S^c(x) \delta(x-y), \tag{A.37}$$

and the hamiltonian

$$H = \tfrac{1}{2} \int (-S_x^a S_x^a + J^a S^a S^a)\, dx. \tag{A.38}$$

It follows from the field equation that $S^a S^a$ is time independent; we can thus set

$$S^a S^a = 1. \tag{A.39}$$

To proceed, we introduce a more compact notation:

$$S \cdot S = S^a S^a \ , \qquad J = \mathrm{diag}\{J^1, J^2, J^3\} \ , \qquad (S\wedge\,)_{ab} = 2\epsilon_{abc} S^c, \tag{A.40}$$

in terms of which the equation reads

$$\frac{dS}{dt} = S \wedge (S_{xx} + JS) = S \wedge \frac{\delta H}{\delta S}. \tag{A.41}$$

The mastersymmetry of degree 1 for this system is [24]

$$\tau_1 = x(S \wedge S_{xx} + S \wedge JS) + S \wedge S_x. \tag{A.42}$$

Quite remarkably, it is hamiltonian with respect to $P_1 \equiv S\wedge$:

$$\tau_1 = S \wedge \frac{\delta B}{\delta S}, \tag{A.43}$$

with B given by

$$B = \int x(-S_x \cdot S_x + S \cdot JS)\ dx, \tag{A.44}$$

which is exactly the first moment of the hamiltonian.

The Landau-Lifshitz equation is integrable and it has an infinite number of conserved integrals H_n (the density of H_n contains n S-factors). The corresponding symmetries are defined by

$$K_n = S \wedge \frac{\delta H_n}{\delta S}. \tag{A.45}$$

These vector fields can be generated recursively from τ_1 by

$$[\tau_1, K_n]_L = \text{const } K_{n+1}, \tag{A.46}$$

which is equivalent to

$$[S \wedge \frac{\delta B}{\delta S}, S \wedge \frac{\delta H_n}{\delta S}]_L = \text{const } S \wedge \frac{\delta H_{n+1}}{\delta S}. \tag{A.47}$$

Using the standard isomorphism between the Lie commutator and the Poisson bracket

$$[S \wedge \frac{\delta B}{\delta S}, S \wedge \frac{\delta H_n}{\delta S}]_L = S \wedge \frac{\delta}{\delta S}(\{B, H_n\}), \tag{A.48}$$

we end up with [24]:

$$\{B, H_n\} = \text{const } H_{n+1}. \tag{A.49}$$

Therefore, the quantum ladder operator of the XYZ chain model survives in the continuous classical limit.

A.5. Higher order mastersymmetries of the classical XX model

Although higher order ladder operators have not been found for the quantum XYZ model, they have simple form in the degenerate case of the XY model, as we have seen in section (6). One can therefore expect the existence of an infinite family of hamiltonian mastersymmetries $\{\tau_n\}$ in the classical continuous limit of the XY model. This is indeed the case, as we now show. To simplify discussion we focus on the XX case.

Using the Jordan-Wigner transformation (9.5), the XX conservation laws (6.7), (6.8) can be written, modulo constants, in the form:

$$H_n^{(\pm)} = \sum_{j\in\Lambda} a_{j+n-1}^+ a_j \mp a_j^+ a_{j+n-1}, \tag{A.50}$$

where the fermionic variables satisfy the anticommutation relations (9.3)-(9.4) (leaving out unnecessary subscripts). To obtain the classical continuum limit we represent the fermionic variables in terms of Grassman (anticommuting) functions as follows:

$$a_{j+n}^+ = \xi^+(x+n\delta), \quad a_{j+n} = \xi(x+n\delta). \tag{A.51}$$

The Poisson bracket is fixed to be:

$$\{\xi(x), \xi^+(y)\} = \{\xi^+(x), \xi(y)\} = \delta(x-y), \tag{A.52}$$

$$\{\xi(x), \xi(y)\} = \{\xi^+(x), \xi^+(y)\} = 0. \tag{A.53}$$

The corresponding classical conservation laws become then (modulo a linear combination of lower order charges):

$$H_n^{(\pm)} = \int [(\partial_x^{n-1}\xi^+)\xi \mp \xi^+(\partial_x^{n-1}\xi)]dx. \tag{A.54}$$

Integrating by parts, one obtains

$$H_n^{(\pm)} = \int [(1 \pm (-)^n)(\partial_x^{n-1}\xi^+)\xi]dx, \tag{A.55}$$

from which one can see that $H_{2n+1}^{(+)}$ and $H_{2n}^{(-)}$ are zero. In other words, even though the XX model has twice as many charges as the nondegenerate XYZ model, half of these charges disappear in the continuous classical limit.

The hamiltonian becomes

$$H_2 = \tfrac{1}{2} H_2^{(+)} = \int \xi^+ \xi_x dx, \tag{A.56}$$

giving the canonical equations of motion:

$$\xi_t = \xi_x, \quad \xi_t^+ = \xi_x^+ \tag{A.57}$$

The degree zero mastersymmetry is easily found to be

$$\tau_0 = \begin{pmatrix} x\xi_x \\ x\xi_x^+ + \xi^+ \end{pmatrix}. \tag{A.58}$$

Indeed, its Lie commutation with the vector field

$$K = \begin{pmatrix} \xi_x \\ \xi_x^+ \end{pmatrix} \tag{A.59}$$

gives $[\tau_0, K]_L = 0$. Higher order mastersymmetries are easily found as

$$\tau_n = \begin{pmatrix} x\partial_x^{n+1}\xi \\ \partial_x^{n+1}(x\xi^+) \end{pmatrix}. \tag{A.60}$$

They satisfy the algebra

$$[\tau_n, \tau_m] = (m-n)\tau_{n+m}. \tag{A.61}$$

Consider the hamiltonian structure P_1:

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A.62}$$

whose components are defined by

$$P_1 \delta(x-y) = \begin{pmatrix} \{\xi(x), \xi(y)\} & \{\xi(x), \xi^+(y)\} \\ \{\xi^+(x), \xi(y)\} & \{\xi^+(x), \xi^+(y)\} \end{pmatrix}. \tag{A.63}$$

All the mastersymmetries in (A.60) are hamiltonian with respect to P_1, with the quantities

$$B_{n+2} = \int x\xi^+ \partial_x^{n+1} \xi dx, \tag{A.64}$$

playing the rôle of the hamiltonians:

$$\tau_n = P_1 \begin{pmatrix} \frac{\delta}{\delta\xi} \\ \frac{\delta}{\delta\xi^+} \end{pmatrix} B_{n+2}. \tag{A.65}$$

A.6. Remark on the classical Heisenberg chain

We conclude this appendix with a remark on the classical lattice Heisenberg model. It is interesting to note that the quantum ladder operator does not survive when only the classical limit is considered. Consider for simplicity the isotropic XXX model, whose classical version is defined by the hamiltonian [52, 53, 54]

$$H = \sum_{j\in\Lambda} \ln(1 + S^a_j S^a_{j+1}), \tag{A.66}$$

with the constraint

$$S^a_j S^a_j = 1. \tag{A.67}$$

This expression for the hamiltonian may look somewhat surprising at first sight. The naive choice would have been $\sum_{j\in\Lambda} S^a_i S^a_{i+1}$, but it turns out to be non-integrable. In particular, the candidate H_3 having the structure of its quantum relative does not commute with this naive form of the hamiltonian. Furthermore, the canonical equation of motion it induces has no zero-curvature representation [47]. On the other hand, recall that the classical limit refers to large values of the spin, and that the integrable higher spin version of the quantum Heisenberg model involves higher powers of $S^a_i S^a_{i+1}$, i.e. for spin s, it is a particular polynomial of degree $2s$ in $S^a_i S^a_{i+1}$ [55]. It thus appears that in the classical $s \to \infty$ limit, the logarithm captures the essential integrability aspect of this polynomial. The lattice Heisenberg hamiltonian (A.66) with the Poisson bracket

$$\{S^a_i, S^b_j\} = 2\epsilon_{abc}\delta_{ij} S^c_j \tag{A.68}$$

defines the classical version of (A.34). Somewhat surprisingly, there is no ladder operator defined with respect to this Poisson structure. To demonstrate this, we introduce

$$B = \sum_{j\in\Lambda} j\ f(1 + S^a_j S^a_{j+1}), \tag{A.69}$$

where f is a function to be determined. By requiring $\{B, H\}$ (with H given by (A.66)) to be independent of j, $f(X)$ is forced to be $\ln(X)$. The candidate H_3 so produced is of the form

$$H_3 = \sum_{j\in\Lambda} \frac{\epsilon_{abc} S^a_j S^b_{j+1} S^c_{j+2}}{(1 + S^d_j S^d_{j+1})(1 + S^e_{j+1} S^e_{j+2})}, \tag{A.70}$$

whose Poisson bracket with H does not vanish.

Appendix B. No-go theorem for the existence of a ladder operator for continuous integrable systems related to the XYZ model

B.1. The nonlinear Schrödinger equation

The first four conservation laws of the quantum nonlinear Schrödinger equation are

$$\begin{aligned} H_0 &= \int \Psi^+\Psi \; dx, \\ H_1 &= \int \Psi_x^+\Psi \; dx, \\ H_2 &= \int (\Psi_x^+\Psi_x + \Psi^+\Psi^+\Psi\Psi)dx, \\ H_3 &= \int (\Psi_{xx}^+\Psi_x + 3\Psi_x^+\Psi^+\Psi\Psi)dx, \end{aligned} \tag{B.1}$$

and the defining commutation relations are

$$[\Psi^+(x), \Psi(y)] = \delta(x-y) \; , \qquad [\Psi(x), \Psi(y)] = 0. \tag{B.2}$$

For definiteness, we treat the repulsive case, and the coupling constant has been rescaled to one.

The problem that we now consider is to try to find an operator B such that, modulo lower order charges,

$$[B, H_n] = c_n \; H_{n+1} \quad \text{for} \quad n = 1, 2, \ldots, \tag{B.3}$$

where the coefficients c_n's are not zero for all n greater then some n_0. Restricting our attention to the four first integrals of motion, which contain terms with at most two creation and two annihilation operators, allows us to reformulate the problem in the two-particle sector without being restrictive.

Recall that a complete set of eigenfunctions in the N-particle sector is given by (see e.g. [56])

$$|\lambda_1, ...\lambda_N\rangle = \frac{1}{\sqrt{N!}} \int \chi_N(x_1, ..., x_N|\lambda_1, ..., \lambda_N)\Psi^+(x_1)...\Psi^+(x_N)|0\rangle, \tag{B.4}$$

where

$$\chi_N = \sum_P (-)^P \prod_{j>k} [\lambda_{P_j} - \lambda_{P_k} - i\mathrm{sgn}(x_j - x_k)] e^{i\sum_{n=1}^N x_n \lambda_{P_n}}. \tag{B.5}$$

P is a permutation of the set $(1, 2, ..., N)$. In the two-particle sector, the expression of the conservation laws $\hat{H}_{1,2,3}$ is (up to irrelevant multiplicative factors)

$$\begin{aligned}\hat{H}_1 =&\partial_1 + \partial_2,\\ \hat{H}_2 =&\partial_1^2 + \partial_2^2 - 2\delta(x_1 - x_2),\\ \hat{H}_3 =&\partial_1^3 + \partial_2^3 - 3\delta(x_1 - x_2)(\partial_1 + \partial_2),\end{aligned} \tag{B.6}$$

with $\partial_i \equiv \partial_{x_i}$.

We now look for a differential operator b such that

$$[\hat{H}_1, b] = c_1\hat{H}_2 \ , \qquad [\hat{H}_2, b] = c_2\hat{H}_3 \ , \tag{B.7}$$

for some constants c_2 and $c_1 \neq 0$. It is simple to see that the first relation is satisfied by the following choice for b:

$$x_1\partial_1^2 + x_2\partial_2^2 - (x_1 - x_2)\delta(x_1 + x_2) \tag{B.8}$$

(this gives $c_1 = 1$). Actually this first commutation relation determines b only up to a term of the form

$$f(x_1 - x_2)M(\partial_1, \partial_2), \tag{B.9}$$

where f is an arbitrary function of the difference $x_1 - x_2$ and M is a general polynomial in the derivatives, which does not contain any x_i dependent terms. The commutator of

$$b = x_1\partial_1^2 + x_2\partial_2^2 - (x_1 - x_2)\delta(x_1 + x_2) + f(x_1 - x_2)M(\partial_1, \partial_2) \tag{B.10}$$

with $\hat{H}_2$ yields

$$[\hat{H}_2, b] = 2(\partial_1^3+\partial_2^3)M-4\delta(\partial_1+\partial_2)+2(\partial_1^2 f)M+2(\partial_1 f)M(\partial_1-\partial_2)+2f(M\delta-\delta M), \tag{B.11}$$

where δ is a shorthand for $\delta(x_1 + x_2)$. We want to write the rhs of the last expression in the form

$$c_2\hat{H}_3 = c_2[\partial_1^3 + \partial_2^3 - 3\delta(x_1 - x_2)(\partial_1 + \partial_2)]. \tag{B.12}$$

This leads to the following equation for the functions f and M:

$$\begin{aligned}2(\partial_1^2 f)M + 2(\partial_1 f)M(\partial_1 - \partial_2) + 2f(M\delta - \delta M) = (c_2 - 2)(\partial_1^3 + \partial_2^3)\\ + (4 - 3c_2)\delta(x_1 - x_2)(\partial_1 + \partial_2).\end{aligned} \tag{B.13}$$

First suppose that $c_2 \neq 2$. Given that the higher order operators on each side of this equation must be the same, under the assumption that $(\partial_1 f) \neq 0$, it forces

$$2(\partial_1 f)M(\partial_1 - \partial_2) = (c_2 - 2)(\partial_1^3 + \partial_2^3). \tag{B.14}$$

Since there is no x_i dependence on the rhs, $(\partial_1 f)$ has to be a constant. With

$$f = a_1(x_1 - x_2) + a_2, \tag{B.15}$$

where by hypothesis $a_1 \neq 0$, M is found to be

$$M = \frac{c_2 - 2}{a_1 - 1}(\partial_1^3 + \partial_2^3)(\partial_1 - \partial_2)^{-1}. \tag{B.16}$$

Equation (B.13) reduces to

$$2f(M\delta - \delta M) = (4 - 3c_2)\delta(\partial_1 + \partial_2). \tag{B.17}$$

With the expressions just obtained for f and M, the last equation is not satisfied. There is thus no solution when $c_2 \neq 2$ and $(\partial_1 f) \neq 0$. A similar argument shows with the assumption $(\partial_1 f) = 0$ there is no solution either. With $c_2 = 2$, (B.13) becomes

$$2(\partial_1^2 f)M + 2(\partial_1 f)M(\partial_1 - \partial_2) + 2f(M\delta - \delta M) = -2\delta(x_1 - x_2)(\partial_1 + \partial_2), \tag{B.18}$$

for which it is easily seen that again there is no solution. This demonstrates the non-existence of an operator b that could satisfy (B.7). In other words, (B.3) is impossible for $c_1 \neq 0$. A similar argument, making use of the Jacobi identity, excludes the possibility that $c_1 = \ldots = c_{n_0 - 1} = 0$ with $c_{n_0} \neq 0$ in (B.3).

B.2. The quantum KdV equation

In terms of the quantum extension of the classical KdV field, denoted by T, the quantum KdV equation reads [23]:

$$\partial_t T = [T, H] \quad , \quad H = \frac{1}{2\pi i} \oint dw \; (TT)(w), \tag{B.19}$$

The parentheses stand for the usual normal ordering [57] i.e.

$$(AB)(w) = \frac{1}{2\pi i} \oint_w \frac{dx}{x - w} A(x)B(w), \tag{B.20}$$

and the subscript w indicates that the contour integration circulates once around the singular point w. The second Poisson structure is now replaced by its quantized form, which we take as the radial operator product expansion [58] (the global form of the Virasoro algebra [59])

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{2\partial_w T(w)}{z-w} + ..., \tag{B.21}$$

where the dots indicate irrelevant regular terms and c is the central charge, a free parameter of the quantum theory that cannot be scaled out. The commutator in (B.19) translates into an operator product expansion

$$[T(z), H] = \frac{1}{2\pi i} \oint_z dw\ T(z)(TT)(w). \tag{B.22}$$

This equation can be written as

$$\partial_t T = \frac{(1-c)}{6}\partial_z^3 T - 3\partial_z(TT). \tag{B.23}$$

The first three conservation laws are [23]

$$\begin{aligned} H_1 &= \oint T\ dz, \\ H_2 &= \oint (TT)\ dz, \\ H_3 &= \oint [(T(TT)) - \frac{(2+c)}{12}(\partial_z T \partial_z T)]\ dz. \end{aligned} \tag{B.24}$$

Following the same strategy as in the nonlinear Schrödinger case, we look for an integral B satisfying (B.3). The first commutation relation is satisfied with the choice

$$B = \oint z(TT)\ dz + B_0, \tag{B.25}$$

where B_0 is in the commutant of H_1. With $B_0 = 0$, it is simple to check that $[B, H_2]$ is not proportional to H_3.

Can we find a non-zero B_0 that commutes with H_1? Clearly $B_0 = \oint z^n A\ dz$ is not in the commutant of H_1 for any $n > 0$. If $n = 0$, the only dimensionally possible choice for a local expression for A is proportional to $\partial_z T$, which makes B_0 vanish. The remaining possibility is to have a non-local expression for A. Because the pole term in the operator product $T(z)A(w)$ has residue $\partial_w A(w)$, we see that this will indeed commute with H_1.

However, in the commutation with H_2, it will produce non-local terms; since there are no such terms in H_3, this last possibility is ruled out.

These arguments can be made even more explicit at $c = -2$ [60]. For this value of the central charge, T can be represented by the bilinear

$$T = (\phi\psi), \tag{B.26}$$

where ϕ and ψ are both fermions of spin 1 with operator product expansion

$$\phi(z)\psi(w) = \frac{-1}{(z-w)^2} \quad , \quad \psi(z)\phi(w) = \frac{1}{(z-w)^2}. \tag{B.27}$$

In terms of these fields, the infinite set of conserved quantities is

$$H_{k/2+1} = \oint dz(\phi^{(k)}\psi)(z), \tag{B.28}$$

where $\phi^{(k)} = \partial_z^k \phi$. For k even these integrals can be reexpressed in terms of T as follows [60, 61]:

$$H_n = \frac{2^{n-1}}{n} \oint dz(\overleftarrow{T^n})(z), \tag{B.29}$$

where

$$(\overleftarrow{T^n}) = (\dots(((TT)T)T)\dots T) \qquad (n \text{ factors}). \tag{B.30}$$

From

$$(TT)(z) = \tfrac{1}{2}\ (\phi''\psi + \phi\psi'')(z), \tag{B.31}$$

where a prime stands for a derivative w.r.t. z, we have

$$B = B_1 + B_0 \qquad \text{with} \quad B_1 = \oint z\phi''\psi dz. \tag{B.32}$$

A simple calculation gives

$$[H_2, B_1] = (2\pi i) \oint w[\phi''\psi''' + \phi^{(5)}\psi](w)\ dw, \tag{B.33}$$

which is clearly not proportional to $H_3 = \oint \phi^{(4)}\psi\ dw$ because

$$[\phi''\psi''' + \phi^{(5)}\phi] \neq \partial_z(\dots). \tag{B.34}$$

At $c = -2$, all conservation laws are bilinear in the two fermionic fields so B_0 is also necessarily bilinear (otherwise non-bilinear terms would be generated from single contractions). The most general non-local and bilinear form for B_0 is

$$B_0 = a \oint z(\partial_z^{-m}\phi)(\partial_z^{m+2}\phi) \ , \qquad a = \text{const.} \tag{B.35}$$

This is indeed in the commutant of H_1. By integration by parts, this can be rewritten in the form

$$B_0 = a(-)^m \oint z\phi''\psi \ dz + a(-)^m(m+2) \oint \phi'\psi \ dz. \tag{B.36}$$

The first term is proportional to B_1, while the second one is simply $H_{3/2}$, which commutes with all H_n. Thus, with the most general form for B, we must still conclude that $[H_2, B]$ cannot be proportional to H_3.

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